

BRANCH RINGS, THINNED RINGS, TREE ENVELOPING RINGS

BY

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ABSTRACT

We develop the theory of “branch algebras”, which are infinite-dimensional associative algebras that are isomorphic, up to taking subrings of finite codimension, to a matrix ring over themselves. The main examples come from groups acting on trees.

In particular, for every field \mathbb{k} we construct a \mathbb{k} -algebra \mathfrak{R} which

- is finitely generated and infinite-dimensional, but has only finite-dimensional quotients;
- has a subalgebra of finite codimension, isomorphic to $M_2(\mathfrak{R})$;
- is prime;
- has quadratic growth, and therefore Gelfand–Kirillov dimension 2;
- is recursively presented;
- satisfies no identity;
- contains a transcendental, invertible element;
- is semiprimitive if \mathbb{k} has characteristic $\neq 2$;
- is graded if \mathbb{k} has characteristic 2;
- is primitive if \mathbb{k} is a non-algebraic extension of \mathbb{F}_2 ;
- is graded nil and Jacobson radical if \mathbb{k} is an algebraic extension of \mathbb{F}_2 .

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1. Introduction

Although rings arising from groups are very interesting from a ring theorists' perspective, they are in a sense "too large", because some proper quotient of them may still contain a copy of the original group. The process of "quotienting out extra material" from a group ring while retaining the original group intact is the "thinning process" described in [43].

In this paper, we consider a natural ring arising from a group acting on a rooted tree, which we call its "tree enveloping ring". This is a re-expression, in terms of matrices, of Said Sidki's construction [43]. If the group's action has some self-similarity modeled on the tree's self-similarity, we may expect the same to happen for the associated ring, and we use this self-similarity as a *leitmotiv* for all our results.

Loosely speaking (see §3.1.6 for a more precise statement), a weakly branch algebra is an algebra \mathfrak{A} such that (1) there is an embedding $\psi: \mathfrak{A} \rightarrow M_d(\mathfrak{A})$ for some d , and (2) for any n there is an element of \mathfrak{A} such that $\psi^n(a)$ has a single non-zero entry. We show (Theorem 3.10) that such algebras may not satisfy a polynomial identity.

The main construction of weakly branch algebras is via groups acting on trees; the algebra \mathfrak{A} is then the linear envelope of the groups' linear representation on the boundary of the tree. We show (Theorem 3.25) that if the groups' orbits on the boundary have polynomial growth of degree d , then the Gelfand–Kirillov dimension of \mathfrak{A} is at most $2d$. In particular, contracting groups generate algebras of finite Gelfand–Kirillov dimension.

We next concentrate in more detail on the rings \mathfrak{A} arising from the group G introduced by Grigorchuk in [21]. Recall that G is a just-infinite, finitely generated torsion group. The algebra \mathfrak{A} over the field \mathbb{F}_2 was already studied by Ana Cristina Vieira in [44]. The following theorem summarizes our results in relation with G :

THEOREM 1.1: *The ring \mathfrak{A} is just-infinite and prime (Theorem 4.3). It is recursively presented (Theorems 4.6 and 4.15), and has quadratic growth (Theorem 4.7 and Corollary 4.18), so its Gelfand–Kirillov dimension is 2. The ring \mathfrak{A} has an ideal \mathfrak{K} , and an embedding $\psi: \mathfrak{A} \rightarrow M_2(\mathfrak{A})$, such that all the following: $\psi^{-1}: M_2(\mathfrak{K}) \rightarrow \mathfrak{K}$, $\mathfrak{K} \rightarrow \mathfrak{A}$, $\psi: \mathfrak{A} \rightarrow M_2(\mathfrak{A})$ are inclusions with finite cokernel¹ (Theorem 4.3).*

Over a field of characteristic 2, the ring \mathfrak{A} is graded (Corollary 4.16), and may

¹ I.e. the image has finite codimension in the target.

be presented as

$$\mathfrak{A} = \langle A, B, C, D \mid A^2, B^2, C^2, B + C + D, BC, CB, DAD, \sigma^n(CACACAC), \sigma^n(DACACAD) \text{ for all } n \geq 0 \rangle,$$

where σ is the substitution $\sigma: \{A, B, C, D\}^* \rightarrow \{A, B, C, D\}^*$ defined by

$$A \mapsto ACA, \quad B \mapsto D, \quad C \mapsto B, \quad D \mapsto C.$$

The subgroup generated by $\{1 + A, 1 + B, 1 + C, 1 + D\}$ is isomorphic to the Grigorchuk group G . The ring \mathfrak{A} also contains a copy of the Laurent polynomials $\mathbb{F}_2[X, X^{-1}]$ (Theorem 4.20).

If the ground field \mathbb{k} has characteristic $\neq 2$, then \mathfrak{A} is semiprimitive. If \mathbb{k} is an algebraic extension of \mathbb{F}_2 , then \mathfrak{A} is graded nil², and its Jacobson radical coincides with its augmentation ideal. If \mathbb{k} is a non-algebraic extension of \mathbb{F}_2 , then \mathfrak{A} is a primitive ring, and is not graded nil (Theorem 4.29).

The following statement summarizes the main properties of the rings constructed:

COROLLARY 1.2: *For any field \mathbb{k} , there is a \mathbb{k} -algebra \mathfrak{R} which*

- *is finitely generated and infinite-dimensional, but has only finite-dimensional quotients;*
- *has a subalgebra of finite codimension, isomorphic to $M_2(\mathfrak{R})$;*
- *is prime;*
- *has quadratic growth, and therefore Gelfand–Kirillov dimension 2;*
- *is recursively presented;*
- *satisfies no identity;*
- *contains a transcendental, invertible element;*
- *is semiprimitive if \mathbb{k} has characteristic $\neq 2$;*
- *is graded if \mathbb{k} has characteristic 2;*
- *is primitive if \mathbb{k} is a non-algebraic extension of \mathbb{F}_2 ;*
- *is graded nil and Jacobson radical if \mathbb{k} is an algebraic extension of \mathbb{F}_2 .*

There are interesting examples of primitive, just-infinite algebras with arbitrary Gelfand–Kirillov dimension [45]; they are constructed by their presentation (as monomial algebras). The present construction proceeds in the opposite direction: the algebras are given as a set of endomorphisms of a vector space, and their algebraic properties are deduced from the representation.

2 I.e. all its homogeneous elements are nil.

1.1. **PLAN.** Section 2 recalls constructions and results concerning groups acting on rooted trees. A few of the results are new (Propositions 2.7 and 2.9); the others are given with brief proofs, mainly to illustrate the parallelism between groups and algebras.

Section 3 introduces branch algebras, and develops general tools and results concerning them; in particular, the branch algebra associated with a group acting on a rooted tree.

Section 4 studies more intricately the branch algebra associated with the Grigorchuk group. Its study then splits in two cases, depending on the characteristic being tame ($\neq 2$) or wild ($= 2$). More results hold in characteristic 2, in particular because the branch algebra is graded; some results hold in both cases but the proofs are simpler in characteristic 2, and therefore are given in greater detail there.

1.2. **NOTATION.** We use the following notational conventions: functions are written $x \mapsto x^f$ if they are part of a group that acts, and $x \mapsto f(x)$ otherwise. Generally groups are written in usual capitals (G), and algebras in gothic (\mathfrak{A}). We use ε for the augmentation map on group rings, $\varpi = \ker \varepsilon$ for the augmentation ideal, and $\text{rad } \mathfrak{A}$ for the Jacobson radical of \mathfrak{A} .

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2. Groups acting on trees

2.1. **GROUPS AND TREES.** We start by reviewing the basic notions associated to groups acting on rooted trees.

2.1.1. *Trees.* Let X be a set of cardinality $\#X \geq 2$, called the **alphabet**. The **regular rooted tree** on X is X^* , the set of (finite) words over X . It admits a natural tree structure by putting an edge between words of the form $x_1 \dots x_n$ and $x_1 \dots x_n x_{n+1}$, for arbitrary $x_i \in X$. The root is then the empty word.

More pedantically, the tree X^* is the Hasse diagram of the free monoid X^* on X , ordered by right divisibility ($v \leq w \Leftrightarrow \exists u : vu = w$).

Let G be a group with given action on a set X . Recall that $A \wr G$, the **wreath product** of A with G , denotes the group $A^X \rtimes G$, or again pedantically the semi-direct product with G of the sections of the trivial A -bundle over X .

2.1.2. Decomposition. Let $W = \text{Aut } X^*$ be the group of graph automorphisms of X^* . For each $n \in \mathbb{N}$, the subset X^n of X^* is stable under W , and is called the n th **layer** of the tree. The group W admits a natural map, called the **decomposition**

$$\phi: W \rightarrow W \wr \mathfrak{S}_X,$$

given by $\phi(g) = (f, \pi_g)$ where $\pi_g \in \mathfrak{S}_X$, the **activity** of g , is the restriction of g to the subset $X \subset X^*$, and $f: X \rightarrow W$ is defined by $x^{\pi_g} w^{f(x)} = (xw)^g$, or in other words $f(x)$ is the compositum $X^* \rightarrow xX^* \xrightarrow{g} \pi_g(x)X^* \rightarrow X^*$, where the first and last arrows are given respectively by insertion and deletion of the first letter.

The decomposition map can be applied, in turn, to each of the factors of $W \wr \mathfrak{S}_X$. By abuse of notation, we say that we iterate the map ϕ on W , yielding $\phi^2: W \rightarrow W \wr \mathfrak{S}_X \wr \mathfrak{S}_X \leq W \wr \mathfrak{S}_{X^2}$, etc. More generally, we write $\phi^n: W \rightarrow W \wr \mathfrak{S}_{X^n}$, and π^n its projection to \mathfrak{S}_{X^n} .

The action of W on X^* uniquely extends, by continuity, to an action on X^ω , the (Cantor) set of infinite sequences over X . The self-similarity of X^ω is expressed via the decomposition $X^\omega = \bigsqcup_{x \in X} X^\omega$. This gives, for all $n \in \mathbb{N}$, a continuous map $X^\omega \rightarrow X^n$ obtained by truncating a word to its first n letters.

2.1.3. X^ -bimodule.* There is a left-action $*$ and a right-action $@$ of the free monoid X^* on W , defined for $x \in X$ and $g \in W$ by

$$x * g: w \mapsto \begin{cases} x(v^g) & \text{if } w = xv, \\ w & \text{otherwise,} \end{cases}$$

$$g @ x: w \mapsto v \text{ if } (xw)^g = x^g v.$$

These actions satisfy the following properties:

- (1) $(g @ v) @ w = g @ (vw), \quad v * (w * g) = (vw) * g,$
- (2) $(gh) @ v = (g @ v)(h @ v^g), \quad v * (gh) = (v * g)(v * h),$
- (3) $g = (v * g) @ v, \quad g = \left(\prod_{v \in X^n} v * (g @ v) \right) \pi_g^n,$

where in the last expression the $v * (g @ v)$ mutually commute when v ranges over the n th layer X^n .

In this terminology, when we wrote the decomposition as $\phi(g) = (f, \pi_g)$, we had $f(x) = g @ x$.

2.1.4. Branchness. Let $G < W$ be a group acting on the regular rooted tree X^* . The **vertex stabilizer** $\text{Stab}_G(v)$ is the subgroup of G fixing $v \in X^*$. The group G is

- level-transitive** if G acts transitively on X^n for all $n \in \mathbb{N}$;
- recursive** if $G @ x < G$ for all $x \in X$;
- weakly recurrent** if it is level-transitive, and $G @ x = G$ for all $x \in X$;
- recurrent** if it is level-transitive, and $\text{Stab}_G(x) @ x = G$ for all $x \in X$;
- weakly branch** if G is level-transitive, and $(v * G) \cap G$ is non-trivial for all $v \in X^*$;
- weakly regular branch** if G is level-transitive, and has a non-trivial normal subgroup K , called the **branching subgroup**, with $x * K < K$ for all $x \in X$;
- branch** if G is level-transitive, and $\langle (v * G) \cap G : v \in X^n \rangle$ has finite index in G for all $n \in \mathbb{N}$;

regular branch if G is level-transitive, and has a finite-index normal subgroup K with $x * K < K$ for all $x \in X$;

Weak branchness can be reformulated in terms of the action on X^ω . Then G is weakly branch if every closed set $F \subsetneq X^\omega$ has a non-trivial fixator $\text{Fix}_G(F) = \{g \in G \mid g(f) = f \forall f \in F\}$.

Note that if G is branch, then K^X has finite index in $\phi(K)$, because it has finite index in G^X and in $G \wr \mathfrak{S}_X$.

Note also that if G is weakly regular branch, then there is a unique maximal branching subgroup K ; it is

$$K = \bigcap_{v \in X^*} (G \cap (v * G)) @ v.$$

PROPOSITION 2.1: *If G is transitive on X and $\text{Stab}_G(x) @ x < G$ for all $x \in X$, then it is recurrent, and therefore its action on $(X^\omega, \text{Bernoulli})$ is ergodic. In particular, G is infinite.*

Proof: Proceed by induction on n . Consider a layer X^n of the tree, and two vertices $x_1 \dots x_n$ and $y_1 \dots y_n$. Since G is branch, it acts transitively on X , so $x_1 \dots x_n$ and $y_1 x_2 \dots x_n$ belong to the same orbit. By induction, $x_2 \dots x_n$ and $y_2 \dots y_n$ are in the same G -orbit; therefore, since $\text{Stab}_G(y_1) = G$, the vertices $y_1 x_2 \dots x_n$ and $y_1 \dots y_n$ belong to the same orbit.

If the action is not ergodic, let $A \subset X^\omega$ be an invariant subset of non- $\{0, 1\}$ measure. Then there exists $n \in \mathbb{N}$ such that $X^\omega \rightarrow X^n$ is not onto; its image is a G -orbit, and thus the action of G is not transitive on the n th layer. ■

PROPOSITION 2.2: *If G is regular branch, then it is regular weakly branch and branch; if it is branch, then it is weakly branch; if it is regular weakly branch, then it is weakly branch.*

Proof: Let G be a regular branch group, with branching subgroup K . By Proposition 2.1, G is infinite so K is non-trivial. This shows that G is regular weakly branch. Assume now only that K is non-trivial, and let $v \in X^n$ be any vertex. Since $K^{X^n} \leq \phi^n(G)$, we may take any $k \neq 1$ in K and consider the element $k * v \in G$. This shows that G is weakly branch. The other implications are of the same nature. ■

Note finally that the group G is determined by a generating set S and the restriction of the decomposition map ϕ to S , in the following sense:

PROPOSITION 2.3: *Let F be a group generated by a set S , and let $\phi: F \rightarrow F \wr \mathfrak{S}_X$ be any map. Then there exists a unique subgroup G of $W = \text{Aut } X^*$ that is generated by S and has decomposition map induced by ϕ through the canonical map $F \rightarrow G$.*

Proof: The decomposition map ϕ yields, by iteration, a map $F \rightarrow \mathfrak{S}_{X^n}$ for all $n \in \mathbb{N}$. This defines an action on the n th layer of the tree X^* , and since they are compatible with each other they define an action of F on the tree. We let G be the quotient of F by the kernel of this action. On the other hand, the action of the generators, and therefore of G , is determined by ϕ , so G is unique. ■

In particular, F may be the free group on S , and ϕ may be simply defined by the choice, for each generator in S , of $\#X$ words and a permutation.

Therefore, in defining a recurrent group, we will only give a list of generators, and their images under ϕ . If $X = \{1, \dots, q\}$, we describe ϕ on generators with the notation

$$\phi(g) = \ll g @ 1, \dots, g @ q \gg \pi_g, \text{ or even } \phi(g) = \ll g_1, \dots, g_q \gg \text{ if } \pi_g = 1,$$

rather than in the form $\phi(g) = (f, \pi)$ with $f(x) = g @ x$.

Note that there may exist other groups G' generated by S , and such that the natural map $F \rightarrow G'$ induces an injective map $G' \rightarrow G' \wr \mathfrak{S}_X$. However, such

G' will not act *faithfully* on X^* . The group G defined by Proposition 2.3 is the *smallest* quotient of F through which the decomposition map factors.

Weakly branch groups G are known to satisfy no identity; i.e. for every $w \neq 1$ in the free group $F(y_1, \dots, y_k)$ there exist $g_1, \dots, g_k \in G$ with $w(g_1, \dots, g_k) \neq 1$. We quote the following general result, due to Miklós Abért:

PROPOSITION 2.4 ([2, Theorem 1]): *Let G be a group acting on a set X , such that for every finite $Y \subset X$ the fixator³ of Y does not fix any point in $X \setminus Y$. Then G does not satisfy any identity.*

His proof goes as follows: let w_i be the length- i prefix of w , and let $x \in X$ be any. Then, inductively on i , one shows that there exist $g = (g_1, \dots, g_k) \in G^k$ such that $x, x^{w_1(g)}, \dots, x^{w_i(g)}$ are all distinct. The following is a weakening of [2, Corollary 4].

COROLLARY 2.5: *If G is weakly branch, then it does not satisfy any identity.*

Proof: Let G act on the boundary X^ω of the tree X^* . Let $Y \subset X$ be a finite subset, and let $\xi \in X \setminus Y$ be arbitrary. Then there exists a vertex $v \in X^*$ on the geodesic ξ but on none of the geodesics in Y . Set $K = G \cap (G * v)$. Since G is weakly branch, K is non-trivial. Assume by contradiction that K fixes ξ . Then since K is invariant under the stabilizer of v , and G acts level-transitively, it follows that K also fixes all images of ξ under the stabilizer of ξ ; this is a dense subset of vX^ω , so K fixes X^ω , which contradicts the non-triviality of K . Therefore there exists $g \in K$ with $g|_Y = 1$ and $\xi^g \neq \xi$, so the conditions of Proposition 2.4 are satisfied. ■

PROPOSITION 2.6 ([29, Lemma 5.4]): *Let G be a weakly branch group. Then its centre is trivial.*

Proof: More generally, take $g \neq 1 \in \text{Aut}(X^*)$; then it moves a vertex u . Since G is weakly branch, there is $h \neq 1$ acting only on the subtree uX^* , and $[g, h] \neq 1$. ■

2.2. DIMENSION. Every countable residually- p group has a representation as a subgroup of $\text{Aut } X^*$, for $X = \{1, \dots, p\}$: fix a descending filtration $G = G_0 \geq G_1 \geq G_2 \geq \dots$ with $\bigcap G_n = \{1\}$ and $[G_n : G_{n+1}] = p$; identify X with G_n/G_{n+1} . Then G/G_n is identified with X^n , and G acts faithfully, by multiplication on cosets, on the tree X^* . In general, this action will not be recurrent.

3 Aka “pointwise stabilizer”.

Moreover, this action may be “inefficient” in that the quotient of G represented by the action on X^n may be quite small — if $G_n \triangleleft G$ this quotient is G/G_n of order p^n , while the largest p -group acting on X^n has order $p^{(p^n-1)/(p-1)}$. This motivates the following definition.

Let $W_n = \pi^n(W)$ be quotient of W acting on X^n . We give W the structure of a compact, totally disconnected metric space by setting

$$d(g, h) = \inf\{1/\#W_n \mid \pi^n(g) = \pi^n(h)\}.$$

We obtain in this way the notion of **closure** and **Hausdorff dimension**. Explicitly, for a subgroup $G < W$, we have by [1]

$$\text{Hdim}(G) = \liminf_{n \rightarrow \infty} \frac{\log \#\pi^n(G)}{\log \#W_n};$$

see also [5]. The Hausdorff dimension of G coincides with that of its closure.

2.2.1. The tree closure. Let $P \leq \mathfrak{S}_X$ be any group acting on X . The **tree closure** of P is the subgroup \overline{P} of W consisting of all $g \in W$ such that $\pi^n(g) \in \imath^n P \leq \mathfrak{S}_{X^n}$ for all $n \in \mathbb{N}$. It is the inverse limit of the groups $\imath^n P$, and is a closed subgroup of W .

We have $\overline{P} = \overline{P} \wr P$, and $\pi^n(\overline{P}) = \pi^{n-1}(\overline{P}) \wr P$, so

$$\#\pi^n(\overline{P}) = (\#\pi^{n-1}(\overline{P}))\#^X \#P,$$

and therefore

$$(4) \quad \#\pi^n(\overline{P}) = (\#P)^{(\#X^n-1)/(\#X-1)}.$$

In particular, $\#W_n = (\#\mathfrak{S}_X)^{(\#X^n-1)/(\#X-1)}$, and \overline{P} has Hausdorff dimension $\log \#P / \log(\#X!)$.

If p is prime and $X = \{1, \dots, p\}$, we will often consider subgroups G of $W_p = \overline{P}$, where $P = \langle (1, 2, \dots, p) \rangle$ is a p -Sylow of \mathfrak{S}_X . The dimension of G will be then computed relative to W_p , by the simple formula

$$\text{Hdim}_p(G) = \frac{\text{Hdim } G}{\text{Hdim } W_p} = \text{Hdim } G \frac{\log(p!)}{\log p}.$$

PROPOSITION 2.7: *Let G be a regular branch group. Then G has positive Hausdorff dimension.*

If furthermore G is a subgroup of W_p , then its relative Hausdorff dimension Hdim_p is rational.

Proof: Let G have branching subgroup K , and for all $n \in \mathbb{N}$ set $G_n = \pi^n(G)$. Let $M \in \mathbb{N}$ be large enough so that $G/\phi^{-2}(K^{X^2})$ maps isomorphically into $G_M/\pi^{M-2}(K^{X^2})$. We then have, for all $n \geq M$,

$$(5) \quad \begin{aligned} \#G_n &= [G : K]\#\pi^n(K) = [G : K][\phi(K) : K^X](\#\pi^{n-1}(K))^{\#X} \\ &= [G : K]^{1-\#X}[\phi(K) : K^X](\#G_{n-1})^{\#X}. \end{aligned}$$

Write $\log \#G_n = \alpha\#X^n + \beta$, for some α, β to be determined; we have, again for $n \geq M$,

$$\alpha\#X^n + \beta = (1 - \#X) \log [G : K] + \log [\phi(K) : K^X] + \#X(\alpha\#X^{n-1} + \beta),$$

so $\beta = \log [G : K] - \log [\phi(K) : K^X]/(\#X - 1)$. Then set

$$\alpha = (\log \#G_M - \beta)/\#X^M.$$

We have solved the recurrence for $\#G_n$, and $\alpha > 0$ because G_n has unbounded order.

Now it suffices to note that $\text{Hdim}(G) = \alpha(\#X - 1)/\log(\#X!)$ to obtain $\text{Hdim}(G) > 0$.

For the last claim, note that all indices in (5) are powers of p , and hence their logarithms in base p are integers. ■

QUESTION 1: Miklós Abért and Bálint Virág [3] show that there exist free subgroups of W of Hausdorff dimension 1. Is there a finitely generated recurrent group of dimension 1? A branch group?

2.3. GROWTH. Let G be a group generated by a finite set S . The **length** of $g \in G$ is defined as $\|g\| = \min\{n \mid g = s_1 \dots s_n \text{ for some } s_i \in S\}$. The **word growth** of G is the function

$$f_{G,S}(n) = \#\{g \in G \mid \|g\| \leq n\}.$$

This function depends on the choice of generating set S . Given $f, g: \mathbb{N} \rightarrow \mathbb{R}$, say $f \lesssim g$ if there exists $M \in \mathbb{N}$ with $f(n) \leq g(Mn)$, and say $f \sim g$ if $f \lesssim g \lesssim f$; then the equivalence class of $f_{G,S}$ is independent of S . The group G has **exponential growth** if $f_{G,S} \sim e^n$, and **polynomial growth** if $f_{G,S} \lesssim n^D$ for some $D \in \mathbb{N}$. In all other cases, $f_{G,S}$ grows faster than any polynomial and slower than any exponential, and G has **intermediate growth**. If furthermore $f_{G,S}(n) \geq A^n$ for some $A > 1$, uniformly on S , then G has **uniformly exponential growth**.

More generally, let E be a space on which G acts, and let $* \in E$ be any. Then the **growth of E** is the function

$$f_{E,*,S}(n) = \#\{e \in E \mid e = g * \text{ with } \|g\| \leq n\}.$$

If $E = G$ with left regular action, we recover the previous definition of growth. We will be interested in the case $E = X^\omega$ with the natural action of G , or equivalently of $E = G/\text{Stab}_G(*)$ for some $* \in X^\omega$.

2.3.1. Contraction. Let G be a finitely generated recurrent group. It is **contracting** if there exist $\lambda < 1$, $n \in \mathbb{N}$ and K such that, for all $g \in G$ and $v \in X^n$, we have $\|g@v\| \leq \lambda\|g\| + K$.

PROPOSITION 2.8 ([12], Proposition 8.11): *If G is contracting, then the growth of $(X^\omega, *)$ is polynomial, of degree at most $-n \log \#X / \log \lambda$.*

*Conversely, if $(X^\omega, *)$ has polynomial growth of degree d , then G is contracting for any n large enough and any $\lambda > (\#X)^{-n/d}$.*

PROPOSITION 2.9: *If G is a finitely generated branch group, and $(X^\omega, *)$ has polynomial growth of degree d , then G has growth*

$$f_G(n) \asymp \exp(n^{d/(d+1)}).$$

Proof: Let us write $q = \#X$. Let K be a branching subgroup, and set $R_0 = \min\{\|g\| \mid g \in K, g \neq 1\}$. Let $n \in \mathbb{N}$ and $v \in X^n$ be given. For $g \in K$ satisfying $\|g\| \leq R_0$, set $h_{v,g} = v * g$. By Proposition 2.8, we have $\|h_{v,g}\| \leq q^{n/d}\|g\| \leq q^{n/d}R_0$.

We now choose for all $v \in X^n$ some $g_v \in K$ with $\|g_v\| \leq R_0$, and consider the corresponding element

$$h = \prod_{v \in X^n} h_{v,g_v}.$$

On the one hand, there are at least 2^{q^n} such elements, because there are at least 2 choices for each g_v . On the other hand, such an element has length at most $q^n q^{n/d} R_0$. If $f(R)$ denotes the growth function of G , we therefore have $f(q^{n+n/d}R_0) \geq 2^{q^n}$, or in other words

$$f(R) \asymp \exp(q^{\log R / (1+1/d) \log q}) = \exp(R^{d/(d+1)}). \quad \blacksquare$$

2.4. MAIN EXAMPLES. The first example of a branch group is W itself, with branching subgroup $K = W$. In this paper, however, we are mainly concerned with countable groups. Assume therefore that X is finite, and choose a section

$\widetilde{\mathfrak{S}}_X$ of $\pi: W \rightarrow \mathfrak{S}_X$, for instance lifting $\rho \in \mathfrak{S}_X$ to $\tilde{\rho}: x_1 \dots x_n \mapsto \rho(x_1)x_2 \dots x_n$. The finitary group \mathfrak{S}_X^b is the subgroup of W generated by the $\phi^{-n}(\widetilde{\mathfrak{S}}_X^{X^n})$, for all $n \in \mathbb{N}$. It is locally finite.

More generally, let P be the lift to W of a transitive subgroup of \mathfrak{S}_X . The finitary closure of P is then the subgroup P^b of W generated by the $\phi^{-n}(P^{X^n})$, for all $n \in \mathbb{N}$. If P is countable, then P^b is a countable subgroup of the tree closure \overline{P} of P .

Much of the interest in branch groups comes from the fact that finitely generated examples exist. The most important ones are:

2.4.1. *The Neumann groups.* Take P a perfect, 2-transitive subgroup of \mathfrak{S}_X , and choose $a, b \in X$. Consider two copies P, \overline{P} of P , and let them act on X^* as follows:

$$(x_1 \dots x_n)^P = (x_1^P)x_2 \dots x_n; \quad (x_1 \dots x_n)^{\overline{P}} = \begin{cases} x_1(x_2 \dots x_n)^P & \text{if } x_1 = a, \\ x_1(x_2 \dots x_n)^{\overline{P}} & \text{if } x_1 = b, \\ x_1 \dots x_n & \text{else.} \end{cases}$$

Let G be the group generated by these two images of P . Then G is a perfect group, studied by Peter Neumann in [35]; it is branch, with branching subgroup $K = G$. Indeed choose $r, s \in P$ with $a^r = a \neq a^s$ and $b^r \neq b = b^s$. Then $\phi[\overline{P}, \overline{P}^r] = P \times 1 \times \dots \times 1$ and $\phi[\overline{P}, \overline{P}^s] = 1 \times \overline{P} \times \dots \times 1$, so $\phi(\overline{P}^P)$ contains $G \times 1 \times \dots \times 1$ and therefore contains $G \times \dots \times G$. Note that P is isomorphic to $\phi(G)/G^X$.

The group G is more simply defined by its decomposition map: G is the unique subgroup of W generated by two copies $P \sqcup \overline{P}$ of P and with decompositions

$$\phi(p) = \ll 1, \dots, 1 \gg p, \quad \phi(\overline{p}) = \ll p, \overline{p}, 1, \dots, 1 \gg,$$

with in the last expression the ‘ p ’ in position a and the ‘ \overline{p} ’ in position b .

The example $P = \text{PSL}_3(2)$, in its action on the 7-point projective plane, was considered in [11], where G was shown to have non-uniformly exponential word growth; see also [46]. These groups are contracting with $n = 1$ and $\lambda = \frac{1}{2}$.

The Hausdorff dimension of G is $\log \#P / \log(\#X!)$, by (4).

2.4.2. *The Grigorchuk group.* This group G acts on the binary tree, with $X = \{1, 2\}$. It is best described as the group generated by $\{a, b, c, d\}$, with given decompositions:

$$(6) \quad \begin{aligned} \phi(a) &= \ll 1, 1 \gg (1, 2), & \phi(b) &= \ll a, c \gg, \\ \phi(c) &= \ll a, d \gg, & \phi(d) &= \ll 1, b \gg. \end{aligned}$$

This group was studied by Rostislav Grigorchuk, who showed in [21] that G is a f.g. infinite torsion group — also known as a “Burnside group”. He then showed in [22] that it has word-growth intermediate between polynomial and exponential; the more precise bounds

$$\exp(n^{0.5157}) \lesssim f_G \lesssim \exp(n^{0.7675})$$

appear in [6, 9]. This group is contracting with $n = 1$ and $\lambda = \frac{1}{2}$.

The group G is a branch group, with branching subgroup $K = \langle [a, b] \rangle^G$ of index 16. Indeed set $x = [a, b]$; then $\phi[x^{-1}, d] = \lll 1, x \ggg$ so $\phi(K)$ contains $K \times K$. Set $x = [a, b]$; then, as a group, K is generated by $\{x, [x, d], [x, d^a]\}$.

The finite quotient $\pi^n(G)$ has order $2^{5 \cdot 2^{n-3} + 2}$, for $n \geq 3$. It follows that G has Hausdorff dimension $5/8$.

Igor Lysinok obtained in [33] a presentation of G by generators and relations:

PROPOSITION 2.10 ([33]): *Consider the endomorphism σ of $\{a, b, c, d\}^*$ defined by*

$$(7) \quad a \mapsto aca, \quad b \mapsto d, \quad d \mapsto c, \quad c \mapsto b.$$

Then

$$(8) \quad G = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, \sigma^n(ad)^4, \sigma^n(adacac)^4 \forall n \geq 0 \rangle.$$

Note that if the relator r is understood as $r = 1$, this gives a ring presentation of the group ring $\mathbb{k}G$. Since the algebra \mathfrak{A} mentioned in the introduction is a quotient of $\mathbb{k}G$, it must have stronger relations than the above — see Theorem 4.6. The last two families of relations, in $\mathbb{k}G$, may be written as $\sigma^n[d^a - 1, d - 1] = 0$ and $\sigma^n[d^{(ac)^2a} - 1, d - 1] = 0$. In essence, these relations are strengthened in \mathfrak{A} to $\sigma^n((d^a - 1)(d - 1)) = 0$ and $\sigma^n((d^{(ac)^2a} - 1)(d - 1)) = 0$ respectively.

2.4.3. *The Gupta–Sidki group.* This group $\bar{\Gamma}$ acts on the ternary tree, with $X = \{1, 2, 3\}$. It is best described as the group generated by $\{x, \gamma\}$, with decompositions

$$\phi(x) = \lll 1, 1, 1 \ggg (1, 2, 3), \quad \phi(\gamma) = \lll \gamma, x, x^{-1} \ggg .$$

This group was studied by Narain Gupta and Said Sidki [25], who showed that $\bar{\Gamma}$ is an infinite 3-torsion group.

This group is contracting with $n = 1$ and $\lambda = \frac{1}{2}$.

The finite quotient $\pi^n(\bar{\Gamma})$ has order $3^{2 \cdot 3^{n-1} + 1}$, for $n \geq 2$. It follows that $\bar{\Gamma}$ has Hausdorff dimension $4/9$ in W_3 .

The group $\bar{\Gamma}$ is branch, with branching subgroup $\bar{\Gamma}' = [\bar{\Gamma}, \bar{\Gamma}]$. Indeed $\phi(\bar{\Gamma}')$ contains $\bar{\Gamma}' \times \bar{\Gamma}' \times \bar{\Gamma}'$, because $\phi([\gamma^{-1}\gamma^{-x^2}, \gamma^x\gamma]) = \ll 1, 1, [x, \gamma] \gg$.

Later Said Sidki constructed a presentation of $\bar{\Gamma}$ by generators and relations [42], and associated an algebra to $\bar{\Gamma}$ — see Theorem 4.1.

2.4.4. Weakly branch groups. Most known examples of recurrent groups are weakly branch. Among those that are not branch, one of the first to be considered acts on the ternary tree $\{1, 2, 3\}^*$:

$$\dot{\Gamma} = \langle x, \delta \rangle \text{ given by } \phi(x) = \ll 1, 1, 1 \gg (1, 2, 3), \quad \phi(\delta) = \ll \delta, x, x \gg .$$

It was studied along with G , $\bar{\Gamma}$ and two other examples in [8, 10]. The finite quotient $\pi^n(\dot{\Gamma})$ has order $3^{\frac{1}{4}(3^n+2n+3)}$, for $n \geq 2$. It follows that $\dot{\Gamma}$ has Hausdorff dimension $1/2$ in W_3 .

Two interesting examples, acting on the binary tree, were also found:

2.4.5. The “BSV” group.

$$G_1 = \langle \tau, \mu \rangle \text{ given by } \phi(\tau) = \ll 1, \tau \gg (1, 2), \quad \phi(\mu) = \ll 1, \mu^{-1} \gg (1, 2);$$

it was studied in [16], where it was shown that it is torsion-free, weakly branch, and constructed a presentation of G_1 . The finite quotient $\pi^{2n}(G_1)$ has order $2^{\frac{1}{3}(2^{2n}-1)+n}$, for $n \geq 1$. It follows that G_1 has Hausdorff dimension $1/3$.

2.4.6. The Basilica group.

$$G_2 = \langle a, b \rangle \text{ given by } \phi(a) = \ll 1, b \gg (1, 2), \quad \phi(b) = \ll 1, a \gg ;$$

it was studied in [24], where it was shown that it is torsion-free and weakly branch, and in [14], where it was shown that it is amenable, though not “sub-exponentially elementary amenable”. The finite quotient $\pi^{2n}(G_2)$ has order $2^{\frac{2}{3}(2^{2n}-1)+n}$, for $n \geq 1$. It follows that G_2 has Hausdorff dimension $2/3$.

All of these groups are contracting with $n = 1$ and $\lambda = 1/\sqrt{2}$.

2.4.7. The odometer. This is a group acting on $\{1, 2\}^*$:

$$\mathbb{Z} = \langle \tau \rangle, \quad \phi(\tau) = \ll 1, \tau \gg (1, 2).$$

Its action on the n th layer is via a 2^n -cycle. It is not weakly branch.

2.4.8. The Lamplighter group. This is the group $G = (\mathbb{Z}/2)^{(\mathbb{Z})} \rtimes \mathbb{Z}$, the semi-direct product with \mathbb{Z} of finitely-supported $\mathbb{Z}/2$ -valued functions on \mathbb{Z} . It acts on $\{1, 2\}^*$:

$$G = \langle a, b \rangle, \quad \phi(a) = \ll a, b \gg (1, 2), \quad \phi(b) = \ll a, b \gg .$$

Again this group is not weakly branch.

3. Algebras

We consider various definitions of “recurrence” and “branchness” in the context of algebras. Let \mathbb{k} be a field, fixed throughout this section.

3.1. ASSOCIATIVE ALGEBRAS. If X is a set, we write $M_X(\mathbb{k}) = M_X$ the matrix algebra of endomorphisms of the vector space $\mathbb{k}X$, and for a \mathbb{k} -algebra \mathfrak{A} we write $M_X(\mathfrak{A}) = M_X(\mathbb{k}) \otimes \mathfrak{A}$.

3.1.1. *Recurrent transitive algebras.* A **recurrent transitive algebra** is an associative algebra \mathfrak{A} , given with an injective homomorphism $\psi: \mathfrak{A} \rightarrow M_X(\mathfrak{A})$, for some set X , such that for every $x, y \in X$ the linear map $\mathfrak{A} \rightarrow M_X(\mathfrak{A}) \rightarrow \mathfrak{A}$, obtained by projecting $\psi(\mathfrak{A})$ on its (x, y) matrix entry, is onto.

The map ψ is called the **decomposition** of \mathfrak{A} , and can be iterated, yielding a map $\psi^n: \mathfrak{A} \rightarrow M_{X^n}(\mathfrak{A})$.

The most naive examples are as follows: consider the vector space $V = \mathbb{k}X^\omega$, and $\mathfrak{A} = \text{End}(V)$. The decomposition map is given by $\psi: a \mapsto (a_{x,y})$ where $a_{x,y}$ is defined on the basis vectors $w \in X^*$ as follows: if $a(xw) = \sum b_v v$, then

$$a_{x,y}(w) = \sum_{v=yv' \in X^\omega} b_v v'.$$

Similarly, consider the vector space $V = \mathbb{k}^{X^\omega}$ of functions on X^ω , and $\mathfrak{A} = \text{End}(V)$. The decomposition map is given by

$$\psi(a) = (a_{x,y}) \quad \text{where } a_{x,y}(f)(w) = a(v \mapsto f(xv))(yw).$$

These examples are meant to illustrate the connection between action on X^ω and recurrent algebras; they will not be considered below. However, all our algebras will be subalgebras of these, i.e. contained in $\otimes^\omega M_X = M_{X^\omega}$.

3.1.2. *Decomposition.* Similarly to Proposition 2.3, a recurrent transitive algebra may be defined by its decomposition map, in the following sense:

LEMMA 3.1: *If \mathfrak{F} is an algebra generated by a set S , and $\psi: \mathfrak{F} \rightarrow M_X(\mathfrak{F})$ is a map such that $\psi(\mathfrak{F})_{x,y} = \mathfrak{F}$ for all $x, y \in X$, then there exists a unique minimal quotient of \mathfrak{F} that is a recurrent transitive algebra.*

Proof: Set $\mathcal{I}_0 = \ker \psi$ and $\mathcal{I}_{n+1} = \psi^{-1}M_X(\mathcal{I}_n)$ for $n \in \mathbb{N}$ and $\mathcal{I} = \bigcup_{n \in \mathbb{N}} \mathcal{I}_n$. Then \mathcal{I} is an ideal in \mathfrak{F} , and \mathfrak{F}/\mathcal{I} is a recurrent transitive algebra. Consider the

ideal \mathfrak{J} generated by all ideals $\mathfrak{K} \leq \mathfrak{F}$ such that $\psi(\mathfrak{K}) \leq M_X(\mathfrak{K})$; then $\mathfrak{J} \leq \mathfrak{I}$, and $\mathfrak{A} = \mathfrak{F}/\mathfrak{J}$ is the required minimal quotient of \mathfrak{F} . ■

It follows that a branch algebra may be defined by a choice, for each generator in S , of $\#X^2$ elements of the free algebra $\mathbb{k}\langle S \rangle$. Note that we do not mention any topology on \mathfrak{A} ; if \mathfrak{A} is to be, say, in the category of C^* -algebras, then the definition becomes much more intricate due to the absence of free objects in that category. The best approach is probably that of a C^* -bimodule considered in [34].

An important feature is missing from the algebras of §3.1.1, namely the existence of finite-dimensional quotients similar to group actions on layers. These are introduced as follows:

3.1.3. Augmented algebras. Let \mathfrak{A} be a recurrent transitive algebra. It is **augmented** if there exists a homomorphism $\varepsilon: \mathfrak{A} \rightarrow \mathbb{k}$, called the **augmentation**, and a subalgebra \mathfrak{P} of M_X with a homomorphism $\zeta: \mathfrak{P} \rightarrow \mathbb{k}$, such that the diagram

$$\begin{array}{ccc}
 \mathfrak{A} & \xrightarrow{\psi} & \psi(\mathfrak{A}) \leq M_X \otimes \mathfrak{A} \\
 \varepsilon \downarrow & & \downarrow 1 \otimes \varepsilon \\
 \mathbb{k} & \xleftarrow{\zeta} & \mathfrak{P}
 \end{array}$$

commutes. We abbreviate “augmented recurrent transitive algebra” to **art algebra**, or **\mathfrak{P} -art algebra** if we wish to emphasize which $\mathfrak{P} \leq M_X$ is used.

Let \mathfrak{P} be a subalgebra of M_X , with augmentation $\zeta: \mathfrak{P} \rightarrow \mathbb{k}$. There are two fundamental examples of art algebras, constructed as follows:

3.1.4. The “tree closure” $\overline{\mathfrak{P}}$. We define for all $n \in \mathbb{N}$ an augmented algebra $\mathfrak{P}_n \leq M_{X^n}$, with $\zeta_n: \mathfrak{P}_n \rightarrow \mathbb{k}$, for $n \in \mathbb{N}$ by $\mathfrak{P}_1 = \mathfrak{P}$, $\zeta_1 = \zeta$, and

$$\mathfrak{P}_{n+1} = \langle m \otimes p \in M_X \otimes \mathfrak{P}_n \mid \zeta_n(p)m \in \mathfrak{P} \rangle.$$

Its augmentation is given by $\zeta_{n+1}(m \otimes p) = \zeta(\zeta_n(p)m)$.

Then there is a natural map $\mathfrak{P}_{n+1} \rightarrow \mathfrak{P}_n$, defined by $m_1 \otimes \cdots \otimes m_{n+1} \mapsto \zeta(m_{n+1})m_1 \otimes \cdots \otimes m_n$. We set $\overline{\mathfrak{P}} = \varprojlim \mathfrak{P}_n$.

Then $\overline{\mathfrak{P}}$ is an art algebra: for $a \in \overline{\mathfrak{P}}$, write $a = \varprojlim a_n$ with $a_n \in \mathfrak{P}_n$. Then $a_{n+1} = \sum m_n \otimes p_n$ with $m_n \in M_X$ and $p_n \in \mathfrak{P}_n$. The sequence m_n is constant equal to m , and we set $\psi(a) = \sum m \otimes \varprojlim p_n$.

The following diagram gives a natural map $\mathfrak{A} \rightarrow \overline{\mathfrak{P}}$ for any \mathfrak{P} -art algebra \mathfrak{A} .

We will always suppose that this map is injective.

$$\begin{array}{ccccccc}
 \mathfrak{A} & \xrightarrow{\psi} & \psi\mathfrak{A} & \xrightarrow{\psi} & \psi^2\mathfrak{A} & \xrightarrow{\psi} & \dots \\
 \varepsilon \downarrow & & \downarrow 1 \otimes \varepsilon & & \downarrow 1 \otimes 1 \otimes \varepsilon & & \\
 \mathbb{k} & \xleftarrow{\zeta} & \mathfrak{P} & \xleftarrow{1 \otimes \zeta} & \mathfrak{P}_2 & \xleftarrow{\dots} & \overline{\mathfrak{P}}
 \end{array}$$

3.1.5. *The “finitary closure”.* This construction starts as above, by noting that the map $\mathfrak{P}_{n+1} \rightarrow \mathfrak{P}_n$, $a_n \otimes p \mapsto \zeta(p)a_n$, is split by $a_n \mapsto a_n \otimes 1$. We let \mathfrak{P}^b be the direct limit of the \mathfrak{P}_n ’s along these inclusions.

Then \mathfrak{P}^b is also an art algebra. Its decomposition is defined on \mathfrak{P}_n as above: $\psi(m \otimes p) = m \otimes p$ for $m \in M_X, p \in \mathfrak{P}_n, m \otimes p \in \mathfrak{P}_{n+1}$.

In some sense, $\overline{\mathfrak{P}}$ is the maximal \mathfrak{P} -art algebra, and \mathfrak{P}^b is a minimal \mathfrak{P} -art algebra. More precisely:

PROPOSITION 3.2: *Let \mathfrak{F} be an augmented algebra generated by a set S , and let $\psi: \mathfrak{F} \rightarrow M_X(\mathfrak{F})$ be a map such that $\psi(\mathfrak{F})_{x,y} = \mathfrak{F}$ for all $x, y \in X$. Set $\mathfrak{P} = \varepsilon\psi(\mathfrak{F}) \leq M_X$, and assume that the augmentation $\varepsilon: \mathfrak{F} \rightarrow \mathbb{k}$ factors to $\zeta: \mathfrak{P} \rightarrow \mathbb{k}$.*

Then there exists a unique art subalgebra \mathfrak{A} of $\overline{\mathfrak{P}}$ that is generated by S and has decomposition map induced by ψ through the canonical map $\mathfrak{F} \rightarrow \mathfrak{A}$.

Proof: For all $n \in \mathbb{N}$ there exists a map $\pi^n = \varepsilon\psi^n: \mathfrak{F} \rightarrow \mathfrak{P}_n$, and these maps are compatible in that $(1 \otimes^n \zeta)\pi^{n+1} = \pi_n$. There is therefore a map $\pi: \mathfrak{F} \rightarrow \overline{\mathfrak{P}}$, and we let \mathfrak{A} be the image of π . This proves the existence part.

Let $\mathfrak{A}' = \mathfrak{F}/\mathfrak{J}'$ be another image of \mathfrak{F} in $\overline{\mathfrak{P}}$. Write $\mathfrak{J} = \ker \pi$. Then by definition of art algebra the images of \mathfrak{A} in \mathfrak{P}_n must be $\pi^n(\mathfrak{F})$, so $\mathfrak{J}' \leq \ker \pi^n$, and $\mathfrak{J}' \leq \mathfrak{J}$. It follows that $\mathfrak{J}' = \mathfrak{J}$, because \mathfrak{A} and \mathfrak{A}' are both contained in $\overline{\mathfrak{P}}$. ■

If $X = \{1, \dots, q\}$, then a maximal augmented subalgebra of M_X is $\mathfrak{P} \cong M_{q-1} \oplus \mathbb{k}$, where the augmentation vanishes on M_{q-1} . The examples we shall consider fall into this class.

For V a vector space, we denote by V° its dual, and we consider $V \otimes V^\circ$ as a subspace of $\text{End}(V)$, under the natural identification $(v \otimes \xi)(w) = \xi(w) \cdot v$.

3.1.6. *Branchness.* Let \mathfrak{A} be a recurrent transitive algebra. We say that \mathfrak{A} is **weakly branch**, if for every $v \in X^*$, writing $|v| = n$, we have

$$\psi^n(\mathfrak{A}) \cap (\mathfrak{A} \otimes (v \otimes v^\circ)) \neq \{0\},$$

where $v \otimes v^\circ$ is the rank-1 projection on $\mathbb{k}v \leq \mathbb{k}X^n$;

weakly regular branch, if there exists a non-trivial ideal $\mathfrak{K} \triangleleft \mathfrak{A}$, called the **branching ideal**, with $M_X(\mathfrak{K}) \leq \psi(\mathfrak{K})$;

branch, if for all $n \in \mathbb{N}$ the ideal $\langle \psi^n(\mathfrak{A}) \cap (\mathfrak{A} \otimes (v \otimes v^\circ)) : v \in X^n \rangle$ has finite codimension in $\psi^n(\mathfrak{A})$;

regular branch, if there exists a finite-codimension ideal $\mathfrak{K} \triangleleft \mathfrak{A}$ with $M_X(\mathfrak{K}) \leq \psi(\mathfrak{K})$.

PROPOSITION 3.3: *Let \mathfrak{A} be an art algebra. Then it is infinite-dimensional.*

If \mathfrak{A} is regular branch, then it is weakly regular branch and branch; if it is branch, then it is weakly branch. If it is weakly regular branch, then it is weakly branch.

Proof: Let \mathfrak{A} be an art algebra; then it is unital. By assumption, the map $\psi_{x,y}: a \mapsto \psi(a)_{x,y}$ is onto. Choose any $x \neq y$; then since $\psi(1)_{x,y} = 0$, so $\psi_{x,y}$ is not one-to-one. It follows that \mathfrak{A} is infinite-dimensional.

Let now \mathfrak{A} be regular branch, with branching ideal \mathfrak{K} . Since \mathfrak{A} is infinite-dimensional, $\mathfrak{K} \neq 0$, so \mathfrak{A} is regular weakly branch. Assume now only that \mathfrak{K} is non-trivial, and let $v \in X^*$ be any vertex. Since $M_{X^n}(\mathfrak{K}) \leq \psi^n(\mathfrak{A})$, we may take any $a \neq 0$ in \mathfrak{K} and consider the element $\psi^{-n}(a \otimes (v \otimes v^\circ)) \neq 0$ in \mathfrak{A} . This shows that \mathfrak{A} is weakly branch. The other implications are of the same nature. ■

The choice of v in the definition of weakly branch algebra may have seemed artificial; the following more general notion is equivalent:

LEMMA 3.4: *Let \mathfrak{A} be a weakly branch algebra. Then for any $n \in \mathbb{N}$ and any $\xi, \eta \in \mathbb{k}X^n$ there exists $a \neq 0$ in \mathfrak{A} with $(1 - P_\xi)(\psi^n a) = 0 = (\psi^n a)(1 - P_\eta)$, where $P_\xi, P_\eta \in M_{X^n}$ denote respectively the projectors on ξ, η .*

Proof: The weakly branch condition amounts to the lemma for $\xi = \eta$ a basis vector (element of X^n) of $\mathbb{k}X^n$. Write in full generality $\xi = \sum \xi_v v$ and $\eta = \sum \eta_w w$, the sums running over $v \in X^n$. Fix $w \in X^n$ and choose $b \neq 0$ with $b \otimes (w \otimes w^\circ) \in \psi^n(\mathfrak{A})$. For all $v, w \in X^n$ choose $c_{v,w}$ with $v\psi^n(c_{v,w}) = w$; this is possible because projection on the (v, w) entry is a surjective map: $\mathfrak{A} \rightarrow \mathfrak{A}$. Finally set

$$a = \sum_{v,w \in X^n} \xi_v c_{v,v_0} b c_{v_0,w} \eta_w. \quad \blacksquare$$

3.2. HAUSDORFF DIMENSION. Let \mathfrak{A} be an art algebra. For every n , it has a representation $\pi^n = \epsilon\psi^n: \mathfrak{A} \rightarrow M_{X^n}(\mathbb{k})$. We define the **Hausdorff dimension** of \mathfrak{A} as

$$\text{Hdim}(\mathfrak{A}) = \liminf_{n \rightarrow \infty} \frac{\dim \pi^n(\mathfrak{A})}{\dim M_{X^n}}$$

Let us compute the Hausdorff dimension of the tree closure $\overline{\mathfrak{P}}$ defined in 3.1.4. There, $\pi^n(\overline{\mathfrak{P}})$ is none other than \mathfrak{P}_n . Let $\varpi_n = \ker \zeta_n$ denote the augmentation ideal of \mathfrak{P}_n . Then, as a vector space, $\mathfrak{P}_{n+1} = M_X \otimes \varpi_n \oplus \mathfrak{P}$, so

$$\dim \mathfrak{P}_{n+1} = \dim(M_X)(\dim \mathfrak{P}_n - 1) + \dim \mathfrak{P}.$$

It follows that

$$\dim \mathfrak{P}_n = \frac{\dim \mathfrak{P} - 1}{\dim M_X - 1} (\dim M_X)^n + \frac{\dim M_X - \dim \mathfrak{P}}{\dim M_X - 1},$$

and since $\dim \mathfrak{P}_0 = 1$ we have

$$\text{Hdim}(\overline{\mathfrak{P}}) = \frac{\dim \mathfrak{P} - 1}{\#X^2 - 1}.$$

If \mathfrak{A} is a \mathfrak{P} -art algebra, we define its **relative Hausdorff dimension** as

$$\text{Hdim}_{\mathfrak{P}}(\mathfrak{A}) = \frac{\text{Hdim}(\mathfrak{A})}{\text{Hdim}(\overline{\mathfrak{P}})} = \text{Hdim}(\mathfrak{A}) \frac{\#X^2 - 1}{\dim \mathfrak{P} - 1}.$$

The following result is an analogue of Proposition 2.7, and is proven along the same lines:

PROPOSITION 3.5: *Let \mathfrak{A} be a regular branch \mathfrak{P} -art algebra. Then $\text{Hdim}_{\mathfrak{P}} \mathfrak{A}$ is a rational number in $(0, 1]$.*

Proof: Let \mathfrak{A} have branching ideal \mathfrak{K} , and for all $n \in \mathbb{N}$ set $\mathfrak{A}_n = \pi^n(\mathfrak{A})$. Let $M \in \mathbb{N}$ be large enough so that $\mathfrak{A}/\psi^{-2}M_{X^2}(\mathfrak{K})$ maps isomorphically into $\mathfrak{A}_M/\pi^{M-2}(\mathfrak{K})$. We then have, for all $n \geq M$,

$$\begin{aligned} \dim \mathfrak{A}_n &= \dim(\mathfrak{A}/\mathfrak{K}) + \dim \pi^n(\mathfrak{K}) \\ &= \dim(\mathfrak{A}/\mathfrak{K}) + \dim(\psi\mathfrak{K}/M_X(\mathfrak{K})) + \#X^2 \dim \pi^{n-1}(\mathfrak{K}) \\ &= (1 - \#X^2) \dim(\mathfrak{A}/\mathfrak{K}) + \dim(\psi\mathfrak{K}/M_X(\mathfrak{K})) + \#X^2 \dim \mathfrak{A}_{n-1}. \end{aligned}$$

We write $\dim \mathfrak{A}_n = \alpha\#X^{2n} + \beta$, for some α, β to be determined; we have

$$\alpha\#X^{2n} + \beta = (1 - \#X^2) \dim(\mathfrak{A}/\mathfrak{K}) + \dim(\psi\mathfrak{K}/M_X(\mathfrak{K})) + \#X^2(\alpha\#X^{2(n-1)} + \beta),$$

so $\beta = \dim(\mathfrak{A}/\mathfrak{K}) - \dim(\psi\mathfrak{K}/M_X(\mathfrak{K})) / (\#X^2 - 1)$. Then set

$$\alpha = (\dim \mathfrak{A}_M - \beta) / \#X^{2M}.$$

We have solved the recurrence for $\dim \mathfrak{A}_n$, and $\alpha > 0$ because \mathfrak{A}_n has unbounded dimension, since \mathfrak{A} is infinite-dimensional by Proposition 3.3.

Now it suffices to note that $\text{Hdim}(\mathfrak{A}) = \alpha$ to obtain $\text{Hdim}_{\mathfrak{P}}(\mathfrak{A}) > 0$. Furthermore, only linear equations with integer coefficients were involved, so $\text{Hdim}(\mathfrak{A})$, and $\text{Hdim}_{\mathfrak{P}}(\mathfrak{A})$, are rational. ■

3.3. TREE ENVELOPING ALGEBRAS. Let G be a recurrent group, acting on a tree X^* . We therefore have a map $\mathbb{k}G \rightarrow \text{End}(\mathbb{k}X^\omega)$, obtained by extending the representation $G \rightarrow \text{Aut } X^\omega$ by \mathbb{k} -linearity to the group algebra. We define the **tree enveloping algebra** of G as the image \mathfrak{A} of the group algebra $\mathbb{k}G$ in $\text{End}(\mathbb{k}X^\omega)$.

This notion was introduced, slightly differently, by Said Sidki in [43]; it has also appeared implicitly in various places, notably [10] and [34].

LEMMA 3.6: *Let \mathfrak{A} be a quotient of the group ring $\mathbb{k}G$, and let $H \leq G$ be a subgroup. Let $\mathfrak{K} \leq \mathfrak{A}$ be the right ideal generated by $\{h - 1 \mid h \in H\}$. Then*

$$\dim \mathfrak{A}/\mathfrak{K} \leq [G : H].$$

Proof: It clearly suffices to prove the claim for $\mathfrak{A} = \mathbb{k}G$. Let $n = [G : H]$ be the index of H in G , and let T be a right transversal of H in G . Given $a \in \mathfrak{A}$, write $a = \sum a(g_i)g_i$ and each $g_i = h_i t_i$ for some $h_i \in H, t_i \in T$. Then we have

$$a = \sum a(g_i)h_i t_i = \sum a(g_i)t_i + \sum a(g_i)(h_i - 1)t_i,$$

so T generates $\mathfrak{A}/\mathfrak{K}$. ■

THEOREM 3.7: *Let G be a recurrent transitive group, and let \mathfrak{A} be its tree enveloping algebra.*

- (1) \mathfrak{A} is an art algebra.
- (2) If G is either a weakly branch group, a regular weakly branch group, a branch group, or a regular branch group, then \mathfrak{A} enjoys the corresponding property.

Proof: Let G be a recurrent transitive group, with decomposition $\phi: G \rightarrow G \wr \mathfrak{S}_X$. Set $\mathfrak{F} = \mathbb{k}G$. We define $\psi: \mathfrak{F} \rightarrow M_X(\mathfrak{F})$ by extending ϕ linearly: for $g \in G$, set

$$\psi(g) = \sum_{x \in X} (g @ x) \otimes (x^g \otimes x^\circ).$$

We also let \mathfrak{P} be the image of $\mathbb{k}\mathfrak{S}_X$ in M_X ; since \mathfrak{S}_X is 2-transitive, $\mathfrak{P} \cong M_{\#X-1} \oplus \mathbb{k}$.

By Proposition 3.2 there is a unique image of \mathfrak{F} that is an art subalgebra of \mathfrak{F} , and by construction this image is \mathfrak{A} .

Assume that G is regular branch, with branching subgroup K . Set

$$\mathfrak{K} = \langle k - 1 : k \in K \rangle.$$

Then \mathfrak{K} is an ideal in \mathfrak{A} , of finite codimension by Lemma 3.6. Since $x * K \leq \phi(K)$, we have $\mathfrak{K} \otimes (x \otimes x^\circ) \leq \psi(\mathfrak{K})$ for all $x \in X$, and since \mathfrak{A} is transitive we get $M_X(\mathfrak{K}) \leq \psi(\mathfrak{K})$, so \mathfrak{A} is regular branch.

Next, assume G is weakly branch, and pick $v \in X^n$. There exists $1 \neq g \in G$ with $g|_{X^\omega \setminus vX^\omega} = 1$, say $g = v * h$. Then $g - 1 \neq 0$, and $0 \neq \psi^n(g - 1) = (h - 1)(v \otimes v^\circ) \in \mathfrak{A} \otimes (v \otimes v^\circ)$, proving that \mathfrak{A} is weakly branch. The other implications are proven similarly. ■

We note that the tree enveloping algebra corresponding to the odometer (§2.4.7) or the lamplighter group (§2.4.8) are isomorphic to their respective group ring. Indeed these groups have a free orbit in their action on X^ω . Branch groups are at the extreme opposite, as we will see below.

QUESTION 2: *If \mathfrak{A} is the tree enveloping algebra of a branch group G , does $\text{Hdim}(G) > 0$ imply $\text{Hdim}(\mathfrak{A}) > 0$? Do we even have $\text{Hdim}_{\mathfrak{p}}(\mathfrak{A}) \geq \text{Hdim}_{\mathfrak{p}}(G)$ for $G \leq W_p$?*

3.3.1. Algebraic properties. Recall that an algebra \mathfrak{A} is **just-infinite** if \mathfrak{A} is infinite-dimensional, and all proper quotients of \mathfrak{A} are finite-dimensional (or, equivalently, all non-trivial ideals in \mathfrak{A} have finite codimension). The **core** of a right ideal $\mathfrak{K} \leq \mathfrak{A}$ is the maximal 2-sided ideal contained in \mathfrak{K} . The **Jacobson radical** $\text{rad } \mathfrak{A}$ is the intersection of the maximal right ideals of \mathfrak{A} . The **upper nil radical** $\text{nil}^* \mathfrak{A}$ is the sum of all nil ideals of \mathfrak{A} .

An algebra \mathfrak{A} is **prime** if, given two non-zero ideals $\mathfrak{J}, \mathfrak{I} \leq \mathfrak{A}$, we have $\mathfrak{J}\mathfrak{I} \neq 0$. It is **primitive** if it has a faithful, irreducible module, or equivalently a maximal right ideal with trivial core. It is **semiprimitive**⁴ if its Jacobson radical is trivial.

LEMMA 3.8: *Let G be a regular branch group, with branching subgroup K . Let \mathfrak{A} be its tree enveloping algebra, with branching ideal \mathfrak{K} . If either $K/[K, K]$ is finite, or G is finitely generated, then $\mathfrak{K}/\mathfrak{K}^2$ is finite-dimensional.*

⁴ Aka *J-semisimple*.

Proof: Consider $\mathfrak{K} = \langle k - 1 : k \in K \rangle \leq \mathbb{k}G$. Then given $k_1, k_2 \in K$ we have

$$[k_1, k_2] - 1 = k_1^{-1}k_2^{-1}((k_1 - 1)(k_2 - 1) - (k_2 - 1)(k_1 - 1)) \in \mathfrak{K}^2,$$

so \mathfrak{K}^2 contains $[K, K] - 1$. This holds *a fortiori* in \mathfrak{A} , so if $K/[K, K]$ is finite the result follows from Lemma 3.6.

If G is finitely generated, then \mathfrak{A} is also finitely generated, so all its finite-codimension subrings are also finitely generated [31]. In particular $\mathfrak{K}/\mathfrak{K}^2$ is finite-dimensional. ■

THEOREM 3.9: *Let \mathfrak{A} be a regular branch tree enveloping algebra. Then any ideal $\mathfrak{J} \leq \mathfrak{A}$ contains $M_{X^n}(\mathfrak{K}^2)$ for some large enough $n \in \mathbb{N}$.*

In particular, if $\mathfrak{K}/\mathfrak{K}^2$ is finite-dimensional, then \mathfrak{A} is just-infinite, and if $\mathfrak{K}^4 \neq 0$, then \mathfrak{A} is prime.

Proof: Assume \mathfrak{A} is the tree enveloping algebra of the group G . Let \mathfrak{J} be a non-trivial ideal of \mathfrak{A} , and choose any non-zero $a \in \mathfrak{J}$. Then $a = \sum a(g_i)g_i$, and the finitely many g_i in the support of a all act differently on X^* . The entries of $\psi^n(a)$, for large enough n , are therefore monomial; more precisely, there exist $v, w \in X^n$ such that the (v, w) entry of $\psi^n(a)$, call it b , is in G , and therefore is invertible.

Since \mathfrak{J} is an ideal, we have for any $v', w' \in X^n$

$$(\mathfrak{K} \otimes (v' \otimes v^\circ))a(\mathfrak{K} \otimes (w \otimes (w')^\circ)) = (\mathfrak{K}b\mathfrak{K}) \otimes (v' \otimes (w')^\circ) \leq \mathfrak{J}.$$

It follows that \mathfrak{J} contains $M_{X^n}(\mathfrak{K}^2)$, which by assumption is cofinite-dimensional.

Assume now that $\mathfrak{J}, \mathfrak{J}'$ are two non-zero ideals of \mathfrak{A} . By the above, there are $n, n' \in \mathbb{N}$ such that \mathfrak{J} contains $M_{X^n}(\mathfrak{K}^2)$ and \mathfrak{J}' contains $M_{X^{n'}}(\mathfrak{K}^2)$. For m larger than $\max\{n, n'\}$ we then have $0 \neq M_{X^m}(\mathfrak{K}^4) \leq \mathfrak{J}\mathfrak{J}'$. ■

Recall also that an algebra \mathfrak{A} is PI (“Polynomial Identity”) if there exists $w \neq 0$ in the free associative algebra $\mathbb{k}\{v_1, \dots, v_k\}$ such that $w(a_1, \dots, a_k) = 0$ for all $a_i \in \mathfrak{A}$. The following result is analogous to 2.5:

THEOREM 3.10: *Let \mathfrak{A} be a weakly branch art algebra. Then it is not PI.*

We prove the theorem using the following result, which may be of independent interest. Let \mathfrak{A} be an algebra acting faithfully on a vector space V . We say that \mathfrak{A} **separates** V if for every finite-dimensional subspace Y of V and any $\xi \notin Y$ there exists $a \in \mathfrak{A}$ with $Ya = 0$ and $\xi a \notin \langle Y, \xi \rangle$.

PROPOSITION 3.11: *Let \mathfrak{A} be an algebra separating a vector space V . Then \mathfrak{A} is not PI.*

Proof: Let $P \in \mathbb{k}\{v_1, \dots, v_k\}$ be a non-commutative polynomial. We will find $a_1, \dots, a_k \in \mathfrak{A}$ and $\eta \in V$ such that $\eta P(a_1, \dots, a_k) \neq 0$. We actually will show more, by induction: let $X_0 \subset \{v_1, \dots, v_k\}^*$ be the set of monomials, without their coefficients, appearing in P , and let X be the set of prefixes of words in X_0 . For any $\eta \neq 0 \in V$, we construct $(a_1, \dots, a_k) \in \mathfrak{A}^k$ such that $\{\eta x(a) \mid x \in X\}$ is an independent family. It then of course follows that $\eta P(a) \neq 0$.

The induction starts with $X = \{1\}$. Then any $\eta \neq 0$ will do. Let now X contain at least two elements, and let $y = v_p \dots v_q v_r$ be a longest element of X . By induction, there exists $a \in \mathfrak{A}^k$ such that $Y_0 = \{\eta x(a) \mid x \in X \setminus \{y\}\}$ is an independent family. If $\eta y(a)$ is linearly independent from Y_0 , we have nothing to do. Otherwise, take $\xi = \eta(v_p \dots v_q)(a)$ and $Y = Y_0 \setminus \{\xi\}$. Since V is separated by \mathfrak{A} , there exists $b \in \mathfrak{A}$ with $Yb = 0$ and $\xi b \notin \langle Y, \xi \rangle$. Set $a'_i = a_i$ for $i \neq r$, and $a'_r = a_r + b$. Then $\{\eta x(a') \mid x \in X\}$ is an independent family. ■

Proof of Theorem 3.10: The algebra \mathfrak{A} is a subalgebra of $\overline{\mathfrak{P}}$, which by definition is a subalgebra of $\varprojlim M_X^{\otimes n}$. We may therefore assume that \mathfrak{A} is a subalgebra of $\text{End}(V)$ for the vector space $V = \varprojlim \mathbb{k}X^n$.

Let Y be a finite-dimensional subspace of V , and let $\xi \notin V$ be arbitrary. Let π^n be the projection $V \rightarrow \mathbb{k}X^n$. Since Y is a closed subspace, there exists $n \in \mathbb{N}$ such that $v = \pi^n(\xi) \notin \pi^n(Y)$, and furthermore such that there is also $w \in \pi^n(V)$ linearly independent from v and $\pi^n(Y)$. By Lemma 3.4 there exists $a \in \mathfrak{A}$ which annihilates Y while it sends v to a multiple of w . Consider all possible such a ; if they all annihilated ξ , then they would also annihilate the orbit of ξ under $P_u \mathfrak{A}$, where $P_u \in M_{X^n}$ denotes projection on u ; since they also annihilate $V(1 - P_u)$, they would all annihilate V , whence $a = 0$ because the representation V is assumed faithful. This contradicts the condition that \mathfrak{A} is weakly branch.

We may therefore apply Proposition 3.11 to conclude that \mathfrak{A} is not PI. ■

In analogy with Proposition 2.6, we have:

PROPOSITION 3.12: *Let \mathfrak{A} be an art algebra which is weakly regular branch, with branching ideal \mathfrak{K} . Assume that \mathfrak{K} is prime. Then $Z(\mathfrak{A}) = 1$.*

Proof: Take $x \in \mathfrak{A}$, and assume that x commutes with \mathfrak{K} ; we wish to show that x is a scalar. For that, write $\psi(x) = (x_{uv})$, and compute $\psi[x, y \otimes (u \otimes v)]$

for all $y \in \mathfrak{K}$ and $u, v \in X$. This matrix vanishes except possibly in its u th row and v th column; the (u, v) -entry is $x_{uu}y - yx_{vv}$, and for $v' \neq v$ and $u' \neq u$ the (u, v') -entry is $yx_{vv'}$ and the (u', v) -entry is $x_{u'u}y$.

If all those entries are to vanish, then $x_{uv}\mathfrak{K} = \mathfrak{K}x_{uv} = 0$ for all $u \neq v$, so $x_{uv} = 0$ because \mathfrak{K} is prime. Similarly $x_{uu} = x_{vv}$ for all u, v , so $\psi(x) = x_{uu} \otimes 1$ for any u . Finally $[x_{uu}, \mathfrak{K}] = 0$, so the argument can be applied to x_{uu} to show that $\psi^n(x)$ is scalar for all n .

Now if x were not scalar there would be $u, v \in X^n$ for some n large enough, such that $x_{uv} \neq 0$ or $x_{uu} \neq x_{vv}$. ■

3.3.2. Compatible filtrations. Let \mathfrak{A} be the tree enveloping algebra of a regular branch group G . We have three descending filtrations of \mathfrak{A} by ideals, namely powers of the branching ideal (\mathfrak{K}^n) , powers of the augmentation ideal (ϖ^n) , and $(M_{X^n}(\mathfrak{K}))$.

PROPOSITION 3.13: *Assume that there is an $n \in \mathbb{N}$ such that $M_{X^n}(\mathfrak{K})$ is contained in \mathfrak{K}^2 . Then the normal subgroups of G control the ideals of \mathfrak{A} : given any non-zero ideal $\mathfrak{J} \leq \mathfrak{A}$, there exists a non-trivial normal subgroup $H \triangleleft G$ with $H - 1 \subset \mathfrak{J}$.*

Proof: By Theorem 3.9, there is $n \in \mathbb{N}$ such that \mathfrak{J} contains $M_{X^{n-1}}(\mathfrak{K}^2)$, so contains $M_{X^n}(\mathfrak{K})$. Set $H = \phi^{-n}(K^{X^n})$; then \mathfrak{J} contains $H - 1$. ■

COROLLARY 3.14: *Assume that there is an $n \in \mathbb{N}$ such that $M_{X^n}(\mathfrak{K})$ is contained in \mathfrak{K}^2 . Then \mathfrak{A} is just-infinite and prime.*

Proposition 3.13 may be used to obtain some information on the Jacobson radical of \mathfrak{A} :

LEMMA 3.15 ([43, Corollary 4.4.3]): *Let \mathbb{k} be a field of characteristic p ; let G be a just-infinite- p group (i.e. an infinite group all of whose proper quotients are finite p -groups), and let \mathfrak{A} be a quotient of $\mathbb{k}G$. Assume that normal subgroups of G control ideals of \mathfrak{A} . Then either $\text{rad } \mathfrak{A} = 0$ or $\text{rad } \mathfrak{A} = \varpi$.*

Proof: $\text{rad } \mathfrak{A} \leq \varpi$ since ϖ is a maximal right ideal. If $\text{rad } \mathfrak{A} \neq 0$, then there is a non-trivial $H \triangleleft G$ with $H - 1 \subset \text{rad } \mathfrak{A}$. Since G/H is a finite p -group, $\mathfrak{A}/\text{rad } \mathfrak{A}$ is a nilpotent algebra, so is 0, and $\text{rad } \mathfrak{A} = \varpi$. ■

This in turn gives control on representations of \mathfrak{A} , by the following result due to Farkas and Small:

PROPOSITION 3.16 ([18]): *Let \mathfrak{A} be a just-infinite, semiprimitive, finitely generated \mathbb{k} -algebra over an uncountable field \mathbb{k} . Then either \mathfrak{A} is primitive, or \mathfrak{A} satisfies a polynomial identity.*

Since weakly branch art algebras satisfy no polynomial identity (Theorem 3.10) they admit irreducible faithful representations as soon as they are semiprimitive.

The following are well known:

PROPOSITION 3.17 ([28, Chapter 4]):

- *If \mathfrak{A} is a just-infinite \mathbb{k} -algebra and contains a transcendental element, then \mathfrak{A} has no non-trivial nil ideal.*
- *If $\text{rad } \mathfrak{A}$ is algebraic, then it is nil.*
- *If \mathfrak{A} is countably generated and \mathbb{k} is uncountable, then the Jacobson radical $\text{rad } \mathfrak{A}$ is nil.*
- *If $x \in \mathfrak{A}$ is transcendental and \mathbb{k} is uncountable, then there exists $\alpha \in \mathbb{k}$ with $1 - \alpha x$ not left-invertible.*

Agata Smoktunowicz has been kind enough to explain the following connection to me:

COROLLARY 3.18: *If \mathfrak{A} is just-infinite, finitely generated over an uncountable field \mathbb{k} , and contains a transcendental element, then \mathfrak{A} is primitive.*

3.3.3. *The tree enveloping algebra of \overline{P} .* Consider as in §2.2.1 a subgroup P of \mathfrak{S}_X , and its tree closure $\overline{P} \leq \text{Aut}(X^*)$. It is regular branch, with branching subgroup \overline{P} .

PROPOSITION 3.19: *Let \mathfrak{A} be the tree enveloping algebra of \overline{P} , and let \mathfrak{P} be the image in M_X of $\mathbb{k}P$. Then $\mathfrak{A} = \overline{\mathfrak{P}}$.*

Proof: Since $\mathfrak{A} \leq \overline{\mathfrak{P}}$, it suffices to show that the natural map $\mathbb{k}\overline{P} \rightarrow \mathfrak{P}_n$ is onto for every n . Let ϖ denote the augmentation ideal of $\mathbb{k}\overline{P}$; then

$$\psi(\mathbb{k}\overline{P}) = M_X(\varpi) + 1 \otimes \mathfrak{P},$$

and therefore $\psi^n(\mathbb{k}\overline{P}) = M_{X^n}(\varpi) + 1 \otimes \mathfrak{P}_n$, and the result follows. ■

The algebra $\overline{\mathfrak{P}}$ can be defined in a different way, following [43]. The group \overline{P} is a profinite (compact, totally disconnected) group, and therefore $\mathbb{k}\overline{P}$ is a topological ring. Consider the ideal

$$(9) \quad \mathfrak{J} = \langle (v * g - 1)(w * h - 1) : v \neq w \in X^n \text{ for some } n; g, h \in \overline{P} \rangle$$

in $\mathbb{k}\overline{P}$. On the one hand, \mathfrak{J} has trivial image in $\overline{\mathfrak{P}}$, since $\psi^n(v * g - 1)$ and $\psi^n(w * h - 1)$ are diagonal matrices with a single non-zero entry, in different coordinates v, w . On the other hand, all relations in the matrix ring $M_{X^n}(\mathbb{k}\overline{P})$ can be reduced to these. It follows that $\overline{\mathfrak{P}}$ equals $\mathbb{k}\overline{P}/\overline{\mathfrak{J}}$, where $\overline{\mathfrak{J}}$, the “thinning ideal”, denotes the closure⁵ of \mathfrak{J} in the topological ring $\mathbb{k}\overline{P}$. More details appear in §4.1.

For any recurrent group G , we may now consider G as a subgroup of some \overline{P} , and therefore $\mathbb{k}G$ is a subalgebra of $\mathbb{k}\overline{P}$. The tree enveloping ring of $\mathbb{k}G$ is then $\mathbb{k}G/(\mathbb{k}G \cap \overline{\mathfrak{J}})$. This was the original definition of tree enveloping rings.

3.4. LIE ALGEBRAS. In this subsection, we let p be a prime, $\mathbb{k} = \mathbb{F}_p$, and fix $X = \{1, \dots, p\}$. Let G be a recurrent subgroup of W_p , with decomposition $\phi: G \rightarrow G \wr C_p$ where C_p is the cyclic subgroup of \mathfrak{S}_X generated by $(1, 2, \dots, p)$. We define the **dimension series** (G_n) of G by $G_1 = G$, and

$$G_n = \langle [g, h]k^p : g \in G, h \in G_{n-1}, k \in G_{\lfloor n/p \rfloor} \rangle.$$

Since G is residually- p , we have $\bigcap G_n = \{1\}$.

The quotient G_n/G_{n+1} is an \mathbb{F}_p -vector space, and we form the “graded group”

$$\text{gr } G = \bigoplus_{n \geq 1} G_n/G_{n+1}.$$

Multiplication and commutation in G endows $\text{gr } G$ with the structure of a graded Lie algebra over \mathbb{F}_p , and $x \mapsto x^p$ induces a Frobenius map on $\text{gr } G$, turning it into a *restricted* Lie algebra.

The dimension series of G can be alternately described, using the augmentation ideal ϖ of \mathbb{F}_pG , as

$$G_n = \{g \in G \mid g - 1 \in \varpi^n\}.$$

Furthermore, consider the graded algebra $\text{gr } \mathbb{F}_pG = \bigoplus_{n \geq 0} \varpi^n/\varpi^{n+1}$ associated to the descending filtration (ϖ^n) of \mathbb{F}_pG . Then

PROPOSITION 3.20 (Lazard [30, Théorème 6.10]; Quillen [41]): *gr \mathbb{F}_pG is the restricted enveloping algebra of $\text{gr } G$.*

3.4.1. Graded tree enveloping algebras. Let \mathfrak{A} be the tree enveloping algebra of the regular branch group G , and assume that \mathfrak{A} is a graded algebra with respect to the filtration (ϖ^n) . Then $\text{gr } G$ embeds isomorphically in \mathfrak{A} .

⁵ Note that [43] does not mention this closure, although it is essential.

PROPOSITION 3.21: *Assume that \mathfrak{A} is a quotient of $\mathfrak{gr} \mathbb{F}_p G$. Then the natural map $\mathfrak{gr} G \hookrightarrow \mathfrak{gr} \mathbb{F}_p G$ induces an embedding $\mathfrak{gr} G \hookrightarrow \mathfrak{A}$.*

Proof: Let $a \in \mathfrak{gr} G$ be such that its image in \mathfrak{A} is trivial. Then, since \mathfrak{A} is graded, all the homogeneous components of a are trivial. But these homogeneous components belong to quotients G_n/G_{n+1} along the dimension series of G , and since $G \hookrightarrow \mathfrak{A}$, they must be trivial in G_n/G_{n+1} . We deduce $a = 0$. ■

If we forget for a moment the distinction between $\mathbb{k}G$ and $\mathfrak{gr} \mathbb{k}G$, Proposition 3.21 can be made more conceptual, by returning to the “thinning process” described after (9): assume G factors as $A \times B$. Then $\mathbb{k}G = \mathbb{k}A \otimes \mathbb{k}B$, and the “thinning” process maps $\mathbb{k}G$ to

$$\mathbb{k}G/\mathfrak{J} = (\mathbb{k}A \oplus \mathbb{k}B)/\{(1, 0) = (0, 1)\},$$

with $\mathfrak{J} = \varpi(\mathbb{k}A) \otimes \varpi(\mathbb{k}B)$. We have $\mathfrak{gr} A \subset \mathbb{k}A$ and $\mathbb{k}B \subset \mathbb{k}B$ and $\mathfrak{gr} G = \mathfrak{gr} A \oplus \mathfrak{gr} B \subset \mathbb{k}G/\mathfrak{J}$. It is in this sense that thinning “respects” Lie elements. More details are given in §4.1.

Proposition 3.21 applies in particular to the group \overline{P} and its tree enveloping algebra $\overline{\mathfrak{P}}$. This points out the recursive structure of $\mathfrak{gr} \overline{P}$, as described in [13].

3.5. GELFAND–KIRILLOV DIMENSION. Let \mathfrak{A} be an algebra (not necessarily associative), with an ascending filtration $(\mathfrak{F}_n)_{n \in \mathbb{Z}}$ by finite-dimensional subspaces. Assume $\mathfrak{F}_n = 0$ for negative n . Then the **Hilbert–Poincaré** series of \mathfrak{A} is the formal power series

$$\Phi_{\mathfrak{A}}(t) = \sum_{n=0}^{\infty} a_n t^n = \sum_{n \geq 0} \dim(\mathfrak{F}_n/\mathfrak{F}_{n-1}) t^n.$$

In particular, if \mathfrak{A} is generated by a finite set S , it has a standard filtration defined as follows: \mathfrak{F}_n is the linear span of all at-most- n -fold products $s_1 \dots s_k$ for all $k \leq n$, in any order (if \mathfrak{A} is not associative).

If $\mathfrak{A} = \bigoplus_{n \geq 0} \mathfrak{A}_n$ is graded, we naturally filter \mathfrak{A} by setting $\mathfrak{F}_n = \mathfrak{A}_0 + \dots + \mathfrak{A}_n$.

If $\dim \mathfrak{F}_n$ grows polynomially, i.e. $p_1(n) \leq \dim \mathfrak{F}_n \leq p_2(n)$ for polynomials p_1, p_2 of the same degree, then \mathfrak{A} has **polynomial growth**. More generally, if $\dim \mathfrak{F}_n$ is bounded from above by a polynomial, the (lower) **Gelfand–Kirillov** dimension of \mathfrak{A} is defined as

$$\text{GKdim}(\mathfrak{A}) = \liminf_{n \rightarrow \infty} \frac{\log \dim \mathfrak{F}_n}{\log n}.$$

If \mathfrak{A} is finitely generated and \mathfrak{F}_n is the span of at-most- n -fold products of generators, then this limit does not depend on the choice of finite generating set.

If \mathfrak{A} is finitely generated and either Lie or associative, then the coefficients a_n may not grow faster than exponentially. A wide variety of intermediate types of growth patterns have been studied by Victor Petrogradsky [38, 39].

Let G be a group, with Lie algebra $\mathfrak{gr} G$. Then the Poincaré–Birkhoff–Witt Theorem gives a basis of $\mathfrak{gr} \mathbb{F}_p G$ consisting of monomials over a basis of $\mathfrak{gr} G$, with exponents at most $p - 1$. As a consequence, we have

PROPOSITION 3.22 (Jennings [26]): *Let G be a group with dimension series (G_n) , and set $\ell_n = \dim_{\mathbb{F}_p}(G_n/G_{n+1})$. Then*

$$\Phi_{\mathfrak{gr} \mathbb{F}_p G}(t) = \prod_{n=1}^{\infty} \left(\frac{1 - t^{pn}}{1 - t^n} \right)^{\ell_n}.$$

Approximations from analytic number theory [32] and complex analysis give then

PROPOSITION 3.23 ([40], Theorem 2.1): *With the notation above for ℓ_n , and $a_n = \dim \varpi^n / \varpi^{n+1}$, we have*

(1) *$\{a_n\}$ grows exponentially if and only if $\{\ell_n\}$ does, and we have*

$$\limsup_{n \rightarrow \infty} \frac{\ln \ell_n}{n} = \limsup_{n \rightarrow \infty} \frac{\ln a_n}{n}.$$

(2) *If $\ell_n \sim n^d$, then $a_n \sim e^{n^{(d+1)/(d+2)}}$.*

A lower bound on the growth of a group G may be obtained from the growth of $\overline{\mathbb{F}_p G}$:

PROPOSITION 3.24 ([23], Lemma 8): *Let G be a group generated by a finite set S , and let $f(n)$ be its growth function. Then*

$$f(n) \geq \dim(\varpi^n / \varpi^{n+1}) \quad \text{for all } n \in \mathbb{N}.$$

It follows that if $\mathfrak{gr} G$ has Gelfand–Kirillov dimension d , then G has growth at least $\exp(n^{(d+1)/(d+2)})$.

It follows that a non-nilpotent residually- p group has growth at least $\exp(\sqrt{n})$. It also follows that 1-relator groups that are not virtually abelian have exponential growth [17].

THEOREM 3.25: *Let G be a contracting group in the sense of §2.3.1, acting on the tree X^* . Let \mathfrak{A} be its tree enveloping algebra. Then \mathfrak{A} has Gelfand–Kirillov dimension*

$$(10) \quad \text{GKdim}(\mathfrak{A}) \leq 2n \frac{\log \#X}{-\log \lambda};$$

*in particular, if $(X^\omega, *)$ has polynomial growth of degree d , then \mathfrak{A} has Gelfand–Kirillov dimension at most $2d$.*

Proof: Let S be the chosen generating set of G , and write $f(r) = \dim_{\mathbb{k}}(\mathbb{k}S^r)$. Then by contraction

$$\mathbb{k}S^r \subset M_{X^n}(\mathbb{k}S^{\lambda r + K}),$$

so $f(r) \leq \#X^{2n} f(\lambda r + K)$. It follows that $\log f(r) / \log r$ converges to the value claimed in (10).

The last remark follows immediately from Proposition 2.8. ■

QUESTION 3: *Assume furthermore that G is branch. Do we then have equality in (10)?*

4. Examples of tree enveloping algebras

We describe here in more detail some tree enveloping algebras. Most of the results we obtain concern the Grigorchuk group. They are modeled on the following result. Said Sidki considers in [43] the tree enveloping algebra \mathfrak{A} of the Gupta–Sidki group $\bar{\Gamma}$ of §2.4.3, over the field \mathbb{F}_3 . He shows:

THEOREM 4.1:

- (1) *The group $\bar{\Gamma}$ and the polynomial ring $\mathbb{F}_3[t]$ embed in \mathfrak{A} .*
- (2) *The algebra \mathfrak{A} is just-infinite, prime, and primitive.*

4.1. THE “THINNING PROCESS”. We recall and generalize the original construction of \mathfrak{A} , since it is relevant to §3.4.1. Let $G \hookrightarrow G \wr P$ be a recurrent group, with $P \leq \mathfrak{S}_X$. Let $\mathfrak{F} = \mathbb{k}G$ be its group algebra. Then we have a natural map

$$\mathfrak{F} \hookrightarrow \mathfrak{F}^{\otimes X} \rtimes P = \mathfrak{F}^{\otimes X} \underline{\otimes} \mathbb{k}P,$$

where $\mathfrak{A} \rtimes P$ designates the crossed product algebra; the $\underline{\otimes}$ indicates the tensor product as vector spaces, with multiplication

$$(1^{\otimes X} \underline{\otimes} \pi)(g_1 \otimes \cdots \otimes g_q \underline{\otimes} 1) = (g_1 \pi \otimes \cdots \otimes g_q \pi \underline{\otimes} 1) \otimes (1^{\otimes X} \underline{\otimes} \pi)$$

for all $g_1, \dots, g_q \in G$ and $\pi \in P$.

We wish to construct a quotient of \mathfrak{F} which still contains a copy of G . For this, let ϖ_i denote, for all $i \in X$, the augmentation ideal of the subalgebra $\mathbb{k} \otimes \dots \otimes \mathfrak{F} \otimes \dots \otimes \mathbb{k} \cong \mathfrak{F}$, with the ‘ \mathfrak{F} ’ in position i ; and let \mathcal{J}_i denote the ideal in $\mathbb{k}P$ generated by $\{\pi - 1 \mid i^\pi = i\}$. Set then

$$\mathfrak{J} = \sum_{i \neq j \in X} \varpi_i \varpi_j \otimes \mathbb{k}P + \sum_{i \in X} \varpi \otimes \mathcal{J}_i + \bigcap_{i \in X} \mathbb{k} \otimes \mathcal{J}_i.$$

LEMMA 4.2 ([43, §3.2]): $\mathfrak{F}/\mathfrak{J} \cong M_X(\mathfrak{F})$.

This process can then be iterated, by thinning the ‘ \mathfrak{F} ’ on the right-hand side of the above; the limit coincides with the tree enveloping ring of G .

4.2. THE GRIGORCHUK GROUP. From now on, we restrict to the Grigorchuk group G defined in §2.4.2. There are two main cases to consider, depending on the characteristic of \mathbb{k} : tame ($\neq 2$) or wild ($= 2$).

We begin by some general considerations. As generating set of G we always choose $S = \{a, b, c, d\}$, and we may again choose S as generating set of its tree enveloping algebra \mathfrak{A} .

Since G ’s decomposition is $\phi: G \mapsto G \wr C_2$, the ring \mathfrak{B} is the linear envelope of the representation of C_2 on two points, i.e. the group ring of C_2 :

$$\mathfrak{B} = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \mid \alpha, \beta \in \mathbb{k} \right\} \cong \mathbb{k}[\mathbb{Z}/2].$$

If \mathbb{k} has characteristic 2, this is the nilpotent ring $\mathbb{k}[t]/(t^2)$; in tame characteristic, $\mathfrak{B} = \mathbb{k} \oplus \mathbb{k}$.

Following Theorem 3.7, we may rewrite G ’s decomposition (6) as a map $\psi: \mathfrak{A} \rightarrow M_2(\mathfrak{A})$:

$$(11) \quad a \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \quad c \mapsto \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad d \mapsto \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.$$

THEOREM 4.3: *The algebra \mathfrak{A} is regular branch, just-infinite, and prime.*

Proof: \mathfrak{A} is regular branch by Theorem 3.7. By Lemma 3.8 and Theorem 3.9 it is just-infinite and prime. ■

Ana Cristina Vieira proved in [44, Corollary 4] that \mathfrak{A} is just-infinite if $\mathbb{k} = \mathbb{F}_2$. Actually her arguments extend to arbitrary characteristic, and also show that \mathfrak{A} is prime.

4.2.1. *Characteristic $\neq 2$.* In this subsection, let \mathbb{k} be a field of characteristic different from 2.

PROPOSITION 4.4: *The algebra \mathfrak{A} is semiprimitive. If furthermore \mathbb{k} is uncountable, then \mathfrak{A} is primitive.*

Proof: The ring \mathfrak{A} admits finite-dimensional quotients $\mathfrak{A}_n = \mathfrak{A}/\mathfrak{P}_n = \pi^n(\mathfrak{A})$. Since \mathbb{k} was assumed of characteristic $\neq 2$ and \mathfrak{A}_n is a quotient of the group algebra of a 2-group, it is semisimple and therefore $\text{rad } \mathfrak{A} \leq \mathfrak{P}_n$ for all n , so $\text{rad } \mathfrak{A} = 0$.

If \mathbb{k} is uncountable, then \mathfrak{A} is primitive by [4, 36]. ■

QUESTION 4: *Is \mathfrak{A} primitive for $\mathbb{k} = \mathbb{Q}$ or \mathbb{F}_p with $p \neq 2$?*

PROPOSITION 4.5: *The algebra \mathfrak{A} has relative Hausdorff dimension $\text{Hdim}_{\mathfrak{P}}(\mathfrak{A})$ equal to 1.*

Proof: This is a reformulation of [10, Theorem 9.7], where the structure of the finite quotient $\pi^n(\mathfrak{A})$ is determined for $\mathbb{k} = \mathbb{C}$. The result obtained was

$$\pi^n(\mathfrak{A}) = \mathbb{C} + \bigoplus_{i=0}^{n-1} M_{2^i}(\mathbb{C}).$$

It follows that $\pi^n(\mathfrak{A})$ has dimension $(4^n + 2)/3$. The proof carries to arbitrary \mathbb{k} of characteristic $\neq 2$. ■

The algebra \mathfrak{A} does not seem to have any natural grading; indeed if ϖ denote the augmentation ideal of \mathfrak{A} , then $\varpi^2 = \varpi$, because ϖ is generated by idempotents $\frac{1}{2}(1 - a), \frac{1}{2}(1 - b), \frac{1}{2}(1 - c), \frac{1}{2}(1 - d)$. As a side note, the Lie powers $\varpi^{[n]}$ of ϖ , defined by $\varpi^{[1]} = \varpi$ and

$$\varpi^{[n+1]} = \mathfrak{A}\{xy - yx \mid x \in \varpi^{[n]}, y \in \varpi\}\mathfrak{A},$$

also seem to stabilize.

The following presentation is built upon Proposition 2.10. Since the proof is similar to that of Theorem 4.15, we only sketch the proof.

THEOREM 4.6: *Consider the endomorphism σ of $\mathbb{k}\langle a, b, c, d \rangle$ defined on its basis by*

$$(12) \quad a \mapsto aca, \quad b \mapsto d, \quad d \mapsto c, \quad c \mapsto b$$

and extended by linearity. Then

$$(13) \quad \mathfrak{A} = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1, \\ \sigma^n((d-1)a(d-1)) = \sigma^n((d-1)a(d^{acac} - 1)) = 0 \ \forall n \geq 0 \rangle.$$

Proof: Let \mathfrak{F} be the free associative algebra on S ; define $\psi: \mathfrak{F} \rightarrow M_2(\mathfrak{F})$ using formulæ (11). Set $\mathfrak{J}_0 = \langle a^2-1, b^2-1, c^2-1, d^2-1, bcd-1 \rangle$, $\mathfrak{J}_{n+1} = \psi^{-1}(M_2(\mathfrak{J}_n))$, and $\mathfrak{J} = \bigcup_{n \geq 0} \mathfrak{J}_n$. We therefore have an algebra $\mathfrak{A}' = \mathfrak{F}/\mathfrak{J}$, and since an easy check shows that the relations above hold in \mathfrak{A} , we have a natural map $\pi: \mathfrak{A}' \rightarrow \mathfrak{A}$ which is onto. We show that it is also one-to-one.

Take $x \in \ker \pi$. Then it is a finite linear combination of words in S^* , so there exists $n \in \mathbb{N}$ such that all entries in $\psi^n(x)$ are linear combinations of words of syllable length at most 1, where a 's and $\{b, c, d\}$'s are grouped in syllables. Since they must also act trivially on $\mathbb{k}X^\omega$, they belong to \mathfrak{J}_0 ; so $x \in \mathfrak{J}_n$.

It remains to compute \mathfrak{J}_n . First, $\mathfrak{J}_1/\mathfrak{J}_0$ is generated by all $(d^u - 1)a(d^v - 1)$ for $u, v \in \{a, b, c, d\}^*$ with an even number of a 's. It is sufficient to consider only $u = 1$; and to assume that v contains only a 's and c 's; indeed d 's can be pulled out to give a shorter relator of the form $(d-1)a(d^w - 1)$, and b 's can be replaced by c 's by the same argument. Using the previous relators, we may then suppose that v is of the form $(ac)^{2k}$.

Next, the relators $r_k = (d-1)a(d^{(ac)^{2k}} - 1) \in \mathfrak{J}_1$ lift to generators $\sigma^n(r_k)$ of $\mathfrak{J}_{n+1}/\mathfrak{J}_n$.

Finally, using the relator $\sigma(r_0) = cacac - aca$, we see that it is sufficient to consider the relators $\sigma^n(r_0)$ and $\sigma^n(r_1)$. ■

Although we may not grade \mathfrak{A} , we may still filter it by powers of the generating set S . We give the following result with minimal proof; it follows from arguments similar to, but harder than, those in Proposition 4.17.

THEOREM 4.7: *The algebra \mathfrak{A} has quadratic growth; therefore its Gelfand-Kirillov dimension is 2.*

More precisely, set $\mathfrak{F}_n = \sum_{i=0}^n \mathbb{k}S^i$ and $a_n = \dim \mathfrak{F}_n/\mathfrak{F}_{n-1}$. Then $a_1 = 4, a_2 = 6, a_3 = 8, a_4 = 10, a_5 = 13, a_6 = 16$, and for $n \geq 7$

$$(14) \quad a_n = \begin{cases} 4n - \frac{3}{2}2^k & \text{if } 2^k \leq n \leq \frac{5}{4}2^k, \\ 3n - \frac{1}{4}2^k & \text{if } \frac{5}{4}2^k \leq n \leq \frac{3}{2}2^k, \\ n + \frac{11}{4}2^k & \text{if } \frac{3}{2}2^k \leq n \leq \frac{7}{4}2^k, \\ 2n + 2^k & \text{if } \frac{7}{4}2^k \leq n \leq 2^{k+1}. \end{cases}$$

It follows for example that, if n is a power of two greater than 4, then

$$\dim \mathfrak{F}_n = \frac{4}{3}n^2 + \frac{5}{4}n + \frac{2}{3}.$$

Note that \mathfrak{A} has Gelfand–Kirillov dimension at most 2, by Theorem 3.25; furthermore, it cannot have dimension 1 since \mathfrak{A} satisfies no polynomial identity by Theorem 3.10, so by Bergman’s gap theorem [27] it has dimension 2.

LEMMA 4.8: *Set $x = ab - ba$ and let $\mathfrak{K} = \mathfrak{A}x\mathfrak{A}$ be the branching ideal of \mathfrak{A} . Then $\mathfrak{A}/\mathfrak{K}$ is 6-dimensional, and $\mathfrak{K}/M_2(\mathfrak{K})$ is 20-dimensional.*

Proof: The codimension of \mathfrak{K} is at most 16, which is the index of K in G . We then check $y = (1 + b)(1 - d) \in \mathfrak{K}$, because $y = \frac{1}{2}(c - 1)xay$, and we use $(d - 1)a(d - 1) = 0$ to see that the codimension of \mathfrak{K} is at most 6, with transversal $\{1, a, d, ad, da, ada\}$. These elements are easily seen to be independent modulo \mathfrak{K} .

The assertion on $\mathfrak{K}/M_2(\mathfrak{K})$ has a similar proof. ■

Sketch of the proof of Theorem 4.7: The first few values of a_n are computed directly. We consider the filtrations $\mathfrak{E}_n = \mathfrak{F}_n \cap \mathfrak{K}$ and $\mathfrak{D}_n = \mathfrak{F}_n \cap M_2(\mathfrak{K})$ of \mathfrak{K} and $M_2(\mathfrak{K})$ respectively. For $n \geq 3$ we have $\dim \mathfrak{F}_n/\mathfrak{E}_n = 6$, and for $n \geq 6$ we have $\dim \mathfrak{E}_n/\mathfrak{D}_n = 20$. It follows that $a_n = \dim \mathfrak{D}_n/\mathfrak{D}_{n-1}$ for n large enough, and we place ourselves in that situation.

A word $w \in S^*$ is **reduced** if it alternates between a -letters and $\{b, c, d\}$ -letters. Every group element in G can be represented by a reduced word. We construct the following refinement of the filtration (\mathfrak{F}_n) : we denote by \mathfrak{F}_n^{at} the linear span of those words $w \in S^*$ which either have length $\leq n - 1$ or are reduced, of length n , start in a , and end in $\{b, c, d\}$. We define similarly $\mathfrak{F}_n^{ta}, \mathfrak{F}_n^{aa}, \mathfrak{F}_n^{tt}$. We set $\mathfrak{E}_n^{at} = \mathfrak{F}_n^{at} \cap \mathfrak{K}$, and define similarly $\mathfrak{E}_n^{ta}, \mathfrak{E}_n^{aa}, \mathfrak{E}_n^{tt}, \mathfrak{D}_n^{at}, \mathfrak{D}_n^{ta}, \mathfrak{D}_n^{aa}, \mathfrak{D}_n^{tt}$. If n is even, then $\mathfrak{D}_n = \mathfrak{D}_n^{at} + \mathfrak{D}_n^{ta}$, while if n is odd, then $\mathfrak{D}_n = \mathfrak{D}_n^{aa} + \mathfrak{D}_n^{tt}$.

The following equalities are not hard to check; the “ \subseteq ” part comes from the contraction of G ’s decomposition map, and the “ \supseteq ” part comes from a construction using the endomorphism σ of (12). For $n \geq 3$ we have

$$\begin{aligned} \mathfrak{D}_{4n} &= \mathfrak{D}_{4n-1} + \mathfrak{D}_{4n}^{at} + \mathfrak{D}_{4n}^{ta} \\ &= \mathfrak{D}_{4n-1} + \begin{pmatrix} \mathfrak{E}_{2n}^{ta} & 0 \\ 0 & \mathfrak{E}_{2n}^{at} \end{pmatrix} + \begin{pmatrix} \mathfrak{E}_{2n}^{at} & 0 \\ 0 & \mathfrak{E}_{2n}^{ta} \end{pmatrix} \\ &= \mathfrak{D}_{4n-1} + \begin{pmatrix} \mathfrak{E}_{2n} & 0 \\ 0 & \mathfrak{E}_{2n} \end{pmatrix}, \end{aligned}$$

and similarly

$$\mathfrak{D}_{4n+1} = \mathfrak{D}_{4n} + \mathfrak{D}_{4n+1}^{aa} + \mathfrak{D}_{4n+1}^{tt} = \mathfrak{D}_{4n} + \begin{pmatrix} 0 & \mathfrak{E}_{2n}^{ta} \\ \mathfrak{E}_{2n}^{at} & 0 \end{pmatrix} + \begin{pmatrix} \mathfrak{E}_{2n+1}^{aa} & 0 \\ 0 & \mathfrak{E}_{2n+1}^{tt} \end{pmatrix},$$

$$\begin{aligned}
 \mathcal{D}_{4n+2} &= \mathcal{D}_{4n+1} + \mathcal{D}_{4n+2}^{at} + \mathcal{D}_{4n+2}^{ta} \\
 &= \mathcal{D}_{4n+1} + \begin{pmatrix} 0 & \mathfrak{E}_{2n+1}^{aa} \\ \mathfrak{E}_{2n+1}^{tt} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathfrak{E}_{2n+1}^{tt} \\ \mathfrak{E}_{2n+1}^{aa} & 0 \end{pmatrix} \\
 &= \mathcal{D}_{4n+1} + \begin{pmatrix} 0 & \mathfrak{E}_{2n+1} \\ \mathfrak{E}_{2n+1} & 0 \end{pmatrix}, \\
 \mathcal{D}_{4n+3} &= \mathcal{D}_{4n+3}^{aa} + \mathcal{D}_{4n+3}^{tt} = \mathcal{D}_{4n+2} + \begin{pmatrix} \mathfrak{E}_{2n+1}^{tt} & 0 \\ 0 & \mathfrak{E}_{2n+1}^{aa} \end{pmatrix} + \begin{pmatrix} 0 & \mathfrak{E}_{2n+2}^{at} \\ \mathfrak{E}_{2n+2}^{ta} & 0 \end{pmatrix}.
 \end{aligned}$$

These equalities give

$$\begin{aligned}
 a_{4n} &= \dim(\mathcal{D}_{4n}/\mathcal{D}_{4n-1}) = 2 \dim(\mathfrak{E}_{2n}/\mathfrak{E}_{2n-1}) = 2a_{2n}, \\
 a_{4n+1} &= \dim(\mathfrak{E}_{2n}^{ta}/\mathfrak{E}_{2n-1}) + \dim(\mathfrak{E}_{2n}^{at}/\mathfrak{E}_{2n-1}) + \dim(\mathfrak{E}_{2n+1}^{aa}/\mathfrak{E}_{2n}) \\
 &\quad + \dim(\mathfrak{E}_{2n+1}^{tt}/\mathfrak{E}_{2n}) = a_{2n} + a_{2n+1}, \\
 a_{4n+2} &= 2 \dim(\mathfrak{E}_{2n+1}/\mathfrak{E}_{2n}) = 2a_{2n+1}, \\
 a_{4n+3} &= \dim(\mathfrak{E}_{2n+1}^{tt}/\mathfrak{E}_{2n}) + \dim(\mathfrak{E}_{2n+1}^{aa}/\mathfrak{E}_{2n}) + \dim(\mathfrak{E}_{2n+2}^{at}/\mathfrak{E}_{2n+1}) \\
 &\quad + \dim(\mathfrak{E}_{2n+2}^{ta}/\mathfrak{E}_{2n+1}) = a_{2n+1} + a_{2n+2},
 \end{aligned}$$

from which (14) follows. ■

4.3. THE GRIGORCHUK GROUP IN CHARACTERISTIC 2. If we let \mathbb{k} be a field of characteristic 2, then sharper results appear. To state them, it is better to choose another generating set for \mathfrak{A} , and throughout this subsection we assume $S = \{A, B, C, D\}$, with $A = a - 1, B = b - 1, C = c - 1, D = d - 1$. In that notation, the augmentation ideal ϖ of \mathfrak{A} is generated by S , and \mathfrak{A} is generated by S as an algebra with one.

We first recall, in a more concrete form, the results stated above for general \mathbb{k} .

PROPOSITION 4.9: *The algebra \mathfrak{A} is recurrent; its decomposition map $\psi : \mathfrak{A} \rightarrow M_2(\mathfrak{A})$ is given by*

$$(15) \quad A \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B \mapsto \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}, \quad C \mapsto \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad D \mapsto \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}.$$

Proof: The expression of ψ follows from the definition. Upon inspection, one sees $1, B, C$ and D in the $(2, 2)$ corner as $\psi(A), \psi(D), \psi(B)$ and $\psi(C)$; then $\psi(ACA + C)$ gives an A in the $(2, 2)$ corner, so projection on the $(2, 2)$ corner is onto. For the other corners, it suffices to multiply the above expressions by $1 + A$ on the left, on the right, or on both sides to obtain all generators in the image of the (i, j) projection. ■

THEOREM 4.10: *The relative Hausdorff dimension of \mathfrak{A} is $\text{Hdim}_{\mathfrak{P}}(\mathfrak{A}) = 7/8$.*

Proof: Let \mathfrak{A}_n be the finite quotient $\pi^n(\mathfrak{A})$ of \mathfrak{A} , and set $b_n = \dim \mathfrak{A}_n$. Then $b_2 = 8$ by direct examination, and one solves the recurrence, for $n \geq 3$,

$$\begin{aligned} b_{n+1} &= \dim \mathfrak{A}_n = \dim \mathfrak{A}/\mathfrak{K} + \dim \pi^{n+1}(\mathfrak{K}) \\ &= 6 + \dim(\mathfrak{K}/M_2(\mathfrak{K})) + 2^2 \dim \pi^n(\mathfrak{K}) = 6 + 8 + 4(b_n - 6) \end{aligned}$$

to $b_n = (14 \cdot 4^{n-2} + 10)/3$. This gives $\text{Hdim}(\mathfrak{A}) = 14/24$, and $\text{Hdim}_{\mathfrak{P}}(\mathfrak{A}) = 7/8$. ■

Let H be the stabilizer in G of the infinite ray $1^\omega \in X^\omega$; then by [10] it is a weakly maximal subgroup, i.e. if $H \not\leq I \leq G$ then I has finite index in G . It follows that the right ideal $\mathfrak{J} = (H - 1)\mathfrak{A}$ is a “weakly maximal” right ideal, i.e. if $\mathfrak{J} \not\leq \mathfrak{J} \leq \mathfrak{A}$ then \mathfrak{J} has finite codimension in \mathfrak{A} . Since the core of \mathfrak{J} is trivial, it follows that \mathfrak{A} admits a faithful module $\mathfrak{A}/\mathfrak{J}$ all of whose quotients are finite. This is none other than the original representation on $\mathbb{k}X^*$.

PROPOSITION 4.11: *The ideal \mathfrak{J} has Gelfand–Kirillov dimension 1; i.e. the dimensions of the quotients $\mathfrak{J} \cap \varpi^n / \mathfrak{J} \cap \varpi^{n+1}$ are bounded.*

Proof: This is a reformulation of [7, Lemma 5.2], where the uniseriality of the modules naturally associated with X^m is proven. ■

From now on, we identify \mathfrak{A} with its image in $M_2(\mathfrak{A})$. We also commit the usual crime of identifying words over S with their corresponding elements in \mathfrak{A} . Set

$$(16) \quad \mathcal{R}_0 = \{A^2, B^2, C^2, D^2, B + C + D, BC, CB, BD, DB, CD, DC, DAD\}.$$

We also set $T = \{B, C, D\}$.

LEMMA 4.12: *All words in \mathcal{R}_0 are trivial in \mathfrak{A} . Furthermore, the last relator is part of a more general pattern: DwD is trivial for any word $w \in S^*$ with $|w| \equiv 1 \pmod 4$.*

Proof: Clearly $A^2 = 0$. Then $B + C + D = \begin{pmatrix} 0 & 0 \\ 0 & B+C+D \end{pmatrix}$ so $B + C + D$ acts trivially on $\mathbb{k}X^\omega$ and is therefore trivial. Given any $x, y \in T$ we have $xy = \begin{pmatrix} 0 & 0 \\ 0 & x'y' \end{pmatrix}$ for some $x', y' \in T$ and these are therefore also relations. Finally, let $w \in S^*$ be a word of length $4n + 1$. Clearly, by the above, $DwD = 0$ unless possibly if w is of the form $Ax_1 \dots Ax_{2n}A$ for some $x_i \in T$. Then

$$w = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{21} \end{pmatrix}$$

where each w_{ij} is a linear combination of words that either start or end in T ; multiplying on both sides with $D = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ therefore annihilates DwD . ■

4.3.1. *A recursive presentation for \mathfrak{A} .* Consider the substitution $\sigma: S^* \rightarrow S^*$, defined as follows:

$$A \mapsto ACA, \quad B \mapsto D, \quad C \mapsto B, \quad D \mapsto C.$$

We say that a word $w \in S^*$ is an $A \div T$ word if its first letter is A and its last letter is in T ; we define similarly $A \div A, T \div A$, and $T \div T$ words. An $\div A$ word is a word ending in A , and $\div T, A \div$ and $T \div$ words are defined similarly.

LEMMA 4.13: *Let $w \in S^*$ represent an element of \mathfrak{K} . Then in \mathfrak{A} we have*

- if w is a $A \div A$ word, then $\sigma(w) = \begin{pmatrix} w & w \\ w & w \end{pmatrix}$;
- if w is a $A \div T$ word, then $\sigma(w) = \begin{pmatrix} 0 & w \\ 0 & w \end{pmatrix}$;
- if w is a $T \div A$ word, then $\sigma(w) = \begin{pmatrix} 0 & 0 \\ w & w \end{pmatrix}$;
- if w is a $T \div T$ word, then $\sigma(w) = \begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix}$, unless w belongs to $\{CAC, CAD, DAC, DAD\}$, in which case $\sigma(w) = \begin{pmatrix} ADA & 0 \\ 0 & w \end{pmatrix}$.

Note in particular that because of the four exceptional cases for $T \div T$ words, the map σ does not induce an endomorphism of \mathfrak{A} . It seems that there does not exist a graded endomorphism τ of \mathfrak{A} with $\psi(\tau(w))_{2,2} = w$ for all long enough $w \in S^*$.

Proof: The induction starts with the words B, CAC, CAD, DAC, DAD . If for example w is a $A \div T$ word, we have $\sigma(w) = \begin{pmatrix} 0 & w \\ 0 & w \end{pmatrix}$, and therefore

$$\sigma(wA) = \begin{pmatrix} 0 & w \\ 0 & w \end{pmatrix} ACA = \begin{pmatrix} 0 & w \\ 0 & w \end{pmatrix} \begin{pmatrix} A+D & A+D \\ A+D & A+D \end{pmatrix} = \begin{pmatrix} wA & wA \\ wA & wA \end{pmatrix},$$

where $wD = 0$ because w ends in a letter in T . ■

PROPOSITION 4.14: *The algebra \mathfrak{A} is regular branch.*

Proof: This follows from Theorem 3.7. Alternatively, consider the ideal

$$\mathfrak{K} = \langle ADA, AB, BA \rangle.$$

Compute $\dim(\mathfrak{A}/\mathfrak{K}) = 6$, with $\mathfrak{A} = \mathfrak{K} \oplus \langle 1, A, B, D, AD, DA \rangle$. Next check

$$\begin{aligned} \begin{pmatrix} ADA & 0 \\ 0 & 0 \end{pmatrix} &= CACAC = C(ADA)C + CA(BA)C \in \mathfrak{K}, \\ \begin{pmatrix} AB & 0 \\ 0 & 0 \end{pmatrix} &= CADA = C(ADA) \in \mathfrak{K}, \\ \begin{pmatrix} BA & 0 \\ 0 & 0 \end{pmatrix} &= ADAC = (ADA)C \in \mathfrak{K}, \end{aligned}$$

giving $M_2(\mathfrak{K}) \leq \mathfrak{K}$. We have $\dim \mathfrak{K}/M_2(\mathfrak{K}) = 8$, because

$$\mathfrak{K} = M_2(\mathfrak{K}) \oplus \langle ADA, AB, BA, ABA, BAB, ABAB, BABA, ABABA \rangle.$$

We may also easily check that $\mathfrak{K}/\mathfrak{K}^2$ is 12-dimensional, by

$$\begin{aligned} \mathfrak{K}^2 = \mathfrak{K} \oplus \langle AB, BA, ABA, ADA, BAB, BAD, DAB, \\ ABAD, ADAB, BADA, DABA, DABAD \rangle. \quad \blacksquare \end{aligned}$$

THEOREM 4.15: *Let \mathcal{R}_0 be as in (16). Then the algebra \mathfrak{A} admits the presentation*

$$\mathfrak{A} = \langle A, B, C, D | \mathcal{R}_0, \sigma^n(CACACAC), \sigma^n(DACACAD) \text{ for all } n \geq 0 \rangle.$$

COROLLARY 4.16: *\mathfrak{A} is graded along powers of its augmentation ideal ω . This grading coincides with that defined by the generating set S .*

Proof: All relations of \mathfrak{A} are homogeneous — they are even all monomial, except for $B + C + D$. \blacksquare

Proof of Theorem 4.15: Let \mathfrak{F} be the free associative algebra on S ; define $\psi: \mathfrak{F} \rightarrow M_2(\mathfrak{F})$ using formulæ (15). Set $\mathfrak{J}_0 = \langle \mathcal{R}_0 \rangle$, $\mathfrak{J}_{n+1} = \psi^{-1}(M_2(\mathfrak{J}_n))$, and $\mathfrak{J} = \bigcup_{n \geq 0} \mathfrak{J}_n$. We therefore have an algebra $\mathfrak{A}' = \mathfrak{F}/\mathfrak{J}$, with a natural map $\pi: \mathfrak{A}' \rightarrow \mathfrak{A}$ which is onto. We show that it is also one-to-one.

Take $x \in \ker \pi$. Then it is a finite linear combination of words in S^* , so there exists $n \in \mathbb{N}$ such that all entries in $\psi^n(a)$ are words in A^* or T^* . Since they must also act trivially on $\mathbb{k}X^\omega$, they belong to \mathfrak{J}_0 ; so $x \in \mathfrak{J}_n$.

It remains to compute \mathfrak{J}_n . First, $\mathfrak{J}_1/\mathfrak{J}_0$ is generated by all DwD with $|w| \equiv 1 \pmod 4$, which map to $0 \in \mathfrak{F}/\mathfrak{J}_0$, and $CACACAC$, which maps to $DAD = 0 \in \mathfrak{F}/\mathfrak{J}_0$. Using the relation $r_0 = DAD$, we see that all DwD are consequences of $r_1 = DACACAD$ and $r_2 = CACACAC$. For example, $DACABAD =$

$r_1 + DACAr_0, r'_1 = DABABAD = DACABAD + r_0ABAD$, and for $n \geq 2$, by induction

$$r'_n = D(AB)^{2n}AD = r'_{n-1}ABAD + r'_{n-2}ABACABAD + D(AB)^{2n-4}A(CABACAr_0 + CAr_1 + r_2AD).$$

Finally, the relations $r_1, r_2 \in \mathfrak{J}_1$ lift to generators $\sigma^n(r_1), \sigma^n(r_2)$ of $\mathfrak{J}_{n+1}/\mathfrak{J}_n$. ■

PROPOSITION 4.17: *Successive powers of the augmentation ideal of \mathfrak{A} satisfy, for $n \geq 3$,*

$$\dim(\varpi^n / \varpi^{n+1}) = \begin{cases} 2n - \frac{1}{2}2^k & \text{if } 2^k \leq n \leq \frac{3}{2}2^k, \\ n + 2^k & \text{if } \frac{3}{2}2^k \leq n \leq 2^{k+1}. \end{cases}$$

It follows that, although $\mathbb{k}G$ has large growth, namely $\dim(\varpi^n / \varpi^{n+1}) \sim \exp(\sqrt{n})$ in $\mathbb{k}G$ by Proposition 3.24, the growth of its quotient \mathfrak{A} is polynomial of degree 2:

COROLLARY 4.18: *The algebra \mathfrak{A} has quadratic growth; therefore its Gelfand–Kirillov dimension is 2, both as a graded algebra (along powers of ϖ), and as a finitely generated filtered algebra.*

Proof of Proposition 4.17: Assume $n \geq 3$. Then we have

$$(17) \quad \varpi^{2n} = \left\langle \varpi^n \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \varpi^n \otimes \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle,$$

$$(18) \quad \varpi^{2n+1} = \left\langle \varpi^n \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \varpi^{n+1} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Indeed consider a generator $w \in S^*$ of ϖ^{2n} . Then w is a word of length $2n$, so is either a $A \div T$ word or a $T \div A$ word. It follows that $\psi(w) = \begin{pmatrix} 0 & u \\ 0 & u \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ u & u \end{pmatrix}$ for some $u \in S^n$, and the ‘ \subseteq ’ inclusion is shown.

Conversely, take $u \in S^n$; if the length of u is even, then u is either a $T \div A$ word or a $A \div T$ word, and set $w = \sigma(u)$. If $|u|$ is odd, then u is either a $T \div T$ word, and consider $w = \sigma(u)A$ and $A\sigma(u)$, or it is a $A \div A$ word, and set $w' = \sigma(u)$ and $w = w'$ with its first or last letter removed. In all cases, w is a word of length $2n$, and $\psi(w) = \begin{pmatrix} 0 & u \\ 0 & u \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ u & u \end{pmatrix}$, which shows the ‘ \supseteq ’ inclusion. A similar argument applies to (18).

Set $a_n = \dim(\varpi^n/\varpi^{n+1})$. Then it is easy to compute

$$\begin{aligned}
 \mathfrak{A}/\varpi &= \langle 1 \rangle && \text{giving } a_0 = 1, \\
 \varpi/\varpi^2 &= \langle A, B, D \rangle && \text{giving } a_1 = 3, \\
 \varpi^2/\varpi^3 &= \langle AB, BA, AD, DA \rangle && \text{giving } a_2 = 4, \\
 \varpi^3/\varpi^4 &= \langle ABA, ADA, BAB, BAD, DAB \rangle && \text{giving } a_3 = 5, \\
 \varpi^4/\varpi^5 &= \langle ABAB, ABAD, ADAB, BABA, BADA, DABA \rangle && \text{giving } a_4 = 6, \\
 \varpi^5/\varpi^6 &= \langle ABABA, ABADA, ADABA, BABAB, \\
 &\quad BABAD, BADAB, DABAB, DABAD \rangle && \text{giving } a_5 = 8,
 \end{aligned}$$

and formulæ (17) and (18) give

$$a_{2n} = 2a_n, \quad a_{2n+1} = a_n + a_{n+1},$$

from which the claim follows. ■

We now show that the filtrations of \mathfrak{A} by (ω^n) , (\mathfrak{K}^n) and $(M_{X^n}(\mathfrak{K}))$ are equivalent:

PROPOSITION 4.19: *For all $n \in \mathbb{N}$ we have*

$$\begin{aligned}
 \varpi^{3n} &\leq \mathfrak{K}^n \leq \varpi^{2n}, \\
 \varpi^{3 \cdot 2^n} &\leq M_{X^n}(\mathfrak{K}) \leq \varpi^{2 \cdot 2^n}.
 \end{aligned}$$

Proof: To check the first assertion, it suffices to note that all non-trivial words of length 3 in S , namely $(AB)A, ADA, (BA)B, (BA)D, D(AB)$, belong to \mathfrak{K} , while all generators of \mathfrak{K} lie in ϖ^3 .

To check the third inclusion, take $w \in S^{3 \cdot 2^n}$; then $\psi^n(w) \in M_{X^n}(\varpi^3)$. To check the fourth inclusion, take a generator w of \mathfrak{K} , and consider $v = \sigma^n(w)$. Since $|w| \geq 2$, we have $|v| \geq 2 \cdot 2^n$ so $v \in \varpi^{2 \cdot 2^n}$. ■

4.3.2. *Laurent polynomials in \mathfrak{A} .* It may seem, since \mathfrak{A} has Gelfand–Kirillov dimension 2, that G contains “most” of the units of \mathfrak{A} . However, G has infinite index in \mathfrak{A}^\times , and contains an element of infinite order:

THEOREM 4.20: *\mathfrak{A} contains the Laurent polynomials $\mathbb{k}[X, X^{-1}]$.*

Proof: Consider the element $X = 1 + A + B + AD$. It is invertible, with

$$X^{-1} = (1 + B)(1 + AC)(1 + ACAC)(1 + A).$$

Now to show that X is transcendental, it suffices to show that X has infinite order; indeed if X were algebraic, it would generate a finite extension of a finite field, and therefore a finite ring; so X would have finite order.

Among words $w \in \{A, B, AD\}^*$, consider the set \mathcal{W} of those of the form

$$w = (AB)^{i_1} AD(AB)^{i_2} AD \dots (AB)^{i_\ell}.$$

These are precisely the words starting by an A , and ending by a B or a D . Define their **length** and **weight** as

$$|w| = \sum_{j=1}^{\ell} (2i_j + 2), \quad \|w\| = \sum_{j=1}^{\ell} (2i_j + 1).$$

Consider the words w_n defined iteratively as follows: $w_1 = ADAB$, and $w_n = \tau(w_{n-1})$ where τ is the substitution $\tau(AB) = (ADAB)^3(AB)^2$, $\tau(AD) = (ADAB)^4$. Then

$$\psi^3(w_n) = w_{n-1} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Define $\sigma(n) = (22 \cdot 8^n - 1)/7$. Then $|w_n| = 4 \cdot 8^n$ and $\|w_n\| = \sigma(n)$; and w_n is the unique summand of $X^{\sigma(n)}$ in \mathcal{W} that belongs to $\varpi^{4 \cdot 8^n}$. This proves that all powers of X are distinct. ■

Note that Georgi Genov and Plamen Siderov show in [20] that $(1+A)(B+C)$, $(1+A)(B+D)$ and $(1+A)(C+D)$ have infinite order in the group ring of G . However, they project to nil-elements in \mathfrak{A} .

Evidently $1 + X$ belongs to the augmentation ideal ϖ , and is also transcendental — in particular, it is not nilpotent. However, ϖ contains many nilpotent elements:

PROPOSITION 4.21 ([44, Theorem 2]): *The semigroup $\{A, B, C, D\}^* \setminus \{1\}$ is nil of degree 8.*

Proof: Let $w \in S^n$ be a semigroup element. If n is odd, then w is either a $T \div T$ word or a $A \div A$ word, so $w^2 = 0$.

If $n \equiv 2 \pmod{4}$, then either w contains a D , in which case $w^2 = 0$ by Lemma 4.12, or $\psi(w^2)$ contains D 's in its non-zero entries, in which case $w^4 = 0$; or $\psi^2(w^4)$ contains D 's in its non-zero entries, in which case $w^8 = 0$.

Finally, if $n \equiv 0 \pmod{4}$, and $n > 0$, then $\psi(w) = \begin{pmatrix} u & v \\ v & v \end{pmatrix}$ or $\begin{pmatrix} u & v \\ u & v \end{pmatrix}$, for some $u, v \in S^{n/2}$ with $uv = vu = 0$. Then $\psi(w^8) = \begin{pmatrix} u^8 & u^8 \\ v^8 & v^8 \end{pmatrix}$ or $\begin{pmatrix} u^8 & v^8 \\ u^8 & v^8 \end{pmatrix}$, and we are done by induction on n . ■

4.3.3. *Nillicity and primitivity of \mathfrak{A} .* To understand the representation theory of \mathfrak{A} , it is important to determine whether \mathfrak{A} is primitive. This depends on the Jacobson radical of \mathfrak{A} , by the following simple result:

PROPOSITION 4.22: *If \mathfrak{A} is semiprimitive, then it is primitive.*

Proof: Since \mathfrak{A} is semiprimitive, $\text{rad } \mathfrak{A} = \bigcap_{\mathfrak{P}} \mathfrak{P} = 0$, where the intersection is taken over all primitive ideals. However, if $\mathfrak{P} \neq 0$ is primitive, then it has finite codimension by Theorem 3.9, so $\mathfrak{A}/\mathfrak{P}$ is finite-dimensional, and therefore nilpotent, because $\mathfrak{A}/\mathfrak{P}$ is the quotient of the group ring of a finite 2-group, so $\mathfrak{P} = \varpi$. The only way to have $\text{rad } \mathfrak{A} = 0$ is therefore that 0 be a primitive ideal.

■

PROPOSITION 4.23: *If \mathbb{k} is a field that is not algebraic over \mathbb{F}_2 , then \mathfrak{A} is primitive.*

Proof: Let t be transcendental over \mathbb{F}_2 , and let $Y = A + B + AD \in \mathfrak{A}(\mathbb{F}_2)$ be transcendental, as in Theorem 4.20. Assume for contradiction that $Y \in \text{rad } \mathfrak{A}$. Then $1 - tY$ is right invertible, i.e. there exists $r \in \mathfrak{A}$ with $(1 - tY)r = 1$. We may assume $r \in \mathfrak{A}(\mathbb{F}_2(t))$, so $(1 - tY)p(t) = q(t)$ for $p(t) \in \mathfrak{A}(\mathbb{F}_2[t])$ and $q(t) \in \mathbb{F}_2[t]$. Again because $1 - tY$ is invertible, we have $p(t) = q(t) \sum_{i=0}^{\infty} t^i Y^i$. Considering this equality in degree higher than $\deg p$ and writing $q(t) = Q(t, 1)$ as a homogeneous polynomial, we get $Q(1, Y) = 0$ whence Y is not transcendental.

Therefore $\text{rad } \mathfrak{A} \neq \varpi$, and $\text{rad } \mathfrak{A} = 0$ by Lemma 3.15, so \mathfrak{A} is primitive by Proposition 4.22. ■

Note that since \mathfrak{A} is primitive for $\mathbb{k} = \mathbb{F}_2(t)$, it has a maximal right ideal L with trivial core, and therefore an irreducible faithful nonprincipal⁶ module \mathfrak{A}/L . One may take any maximal ideal L containing $(1 - tY)\mathfrak{A}$ with Y as in the proof of Proposition 4.23; however, there does not seem to be any handy construction of such an L . On the other hand, the arguments in [37, §2] show that there are infinitely many nonprincipal irreducible representations of \mathfrak{A} .

LEMMA 4.24 (A. Smoktunowicz): *Let \mathfrak{J} be a graded algebra (without unit) generated in degree 1. Then the following are equivalent:*

- (1) \mathfrak{J} is Jacobson radical;
- (2) $M_n(\mathfrak{J})$ is graded nil^l for all n ;

6 I.e. not of the form $e\mathfrak{A}$ for an idempotent e . Since \mathfrak{A} is graded with 1-dimensional degree-0 component, it has no idempotent except 0 and 1.

7 $M_n(\mathfrak{A})$ is naturally \mathbb{Z}^{n^2} -graded by grading all entries independently.

(3) $M_n(\mathfrak{J}_1)$ is nil for all n , where \mathfrak{J}_1 denotes the degree-1 component of \mathfrak{J} .

Proof: We denote by \mathfrak{F} the algebra \mathfrak{J} with a unit adjoined. If \mathfrak{J} is Jacobson radical, then $M_n(\mathfrak{J})$ is radical for all n . Take $x \in M_n(\mathfrak{F})$, homogeneous of degree d . Then, since $1 - x \in M_n(\mathfrak{F})$ is invertible, the sum $\sum_{i \geq 0} x^i$ must converge; now the component of degree di of this sum is x^i ; therefore $x^i = 0$ for i large enough.

The next implication is obvious.

Finally, assume $M_n(\mathfrak{J})$ is graded nil for all n , and choose $x \in \mathfrak{J}$; write $x = x_1 + \dots + x_r$ as a sum of monomials. Furthermore, write each monomial x_i , of degree d_i , as a product $x_i = x_{i,1} \dots x_{i,d_i}$ of monomials of degree 1. Set

$$A = \{(i, j) \mid 1 \leq i \leq r, 1 \leq j < d_i\} \cup \{(0, 0)\}.$$

Construct the matrix $X \in M_A(\mathfrak{J}_1)$ by

$$X_{(i,j),(i',j')} = \begin{cases} x_{i,j+1} & \text{if } i = i' \text{ and } j + 1 = j', \\ x_{i',1} & \text{if } (i, j) = (0, 0) \text{ and } j' = 1, \\ x_{i,d_i} & \text{if } j = d_i - 1 \text{ and } (i', j') = (0, 0), \\ \sum_{k:d_k=1} x_k & \text{if } (i, j) = (i', j') = (0, 0). \end{cases}$$

Since $M_n(\mathfrak{J}_1)$ is nil, there exists $N \in \mathbb{N}$ such that $X^N = 0$. Now write formally $(1 - x)^{-1} = 1 + y_1 + y_2 + \dots$ as a sum of homogeneous components. Then by induction $(X^s)_{(0,0),(i,j)} = y_{s-j} x_{i,1} \dots x_{i,j}$ if $i \geq 1$ and $s > j$, and $y_s = (X^s)_{(0,0),(0,0)}$; therefore $y_s = 0$ as soon as $s \geq N$, and $(1 - x)^{-1}$ exists, so $x \in \text{rad } \mathfrak{J}$ and \mathfrak{J} is Jacobson radical. ■

LEMMA 4.25: *The algebra \mathfrak{A} is graded nil if and only if $M_n(\mathfrak{A})$ is graded nil for all n .*

Proof: Choose a homogeneous element $x \in M_n(\mathfrak{A})$ of degree ≥ 1 . It costs nothing to assume that n is a power of two, say $n = 2^t$. Then since $\omega^3 \leq \mathfrak{K}$ by Proposition 4.19, we have $x^3 \in M_n(\mathfrak{K})$, and therefore $y = \psi^{-t}(x^3) \in \mathfrak{K}$ is homogeneous. It follows that y , and therefore x , are nil elements. ■

PROPOSITION 4.26: *The algebra \mathfrak{A} is non-primitive if and only if it is graded nil (i.e. all homogeneous elements of degree ≥ 1 are nil).*

Proof: Assume first that \mathfrak{A} is not graded nil. Then \mathfrak{A} is not Jacobson radical by Lemma 4.24, so $\text{rad } \mathfrak{A} = 0$ by Lemma 3.15, and \mathfrak{A} is primitive by Proposition 4.22.

Assume next that \mathfrak{A} is graded nil. Then $M_n(\mathfrak{A})$ is graded nil by Lemma 4.25. By Lemma 4.24 the ideal ϖ is Jacobson radical, so $\text{rad } \mathfrak{A} = \varpi$. ■

We denote below by \mathfrak{A}_n the homogeneous part of \mathfrak{A} of degree n . The products $\mathfrak{A}_{n_1} \cdots \mathfrak{A}_{n_k}$ etc. are to be understood as setwise products, and not linear spans of products.

LEMMA 4.27: *Let $n = n_1 + \cdots + n_k$ be even. Assume that for any choice of n'_i, n''_i such that $|n_i - 2n'_i| \leq 1$ and $|n_i - 2n''_i| \leq 1$ and $\sum n'_i + n''_i = n$ we have*

$$(\mathfrak{A}_{n'_1} \mathfrak{A}_{n'_2} \cdots \mathfrak{A}_{n'_k} \mathfrak{A}_{n''_1} \mathfrak{A}_{n''_2} \cdots \mathfrak{A}_{n''_k})^t = 0.$$

Then

$$(\mathfrak{A}_{n_1} \mathfrak{A}_{n_2} \cdots \mathfrak{A}_{n_k})^{2t+1} = 0.$$

Proof: Choose $w_i \in \mathfrak{A}_{n_i}$, and write $w = w_1 \cdots w_k$. For those n_i which are even, we can write $w_i = w'_i + w''_i$ with w'_i a linear combination of $A \div T$ words and w''_i a linear combination of $T \div A$ words, while for the odd n_i we can write $w_i = w'_i + w''_i$ with w'_i a linear combination of $A \div A$ words and w''_i a linear combination of $T \div T$ words.

We next switch the w'_i and w''_i so that w'_1 is a $A \div$ word, and w'_{i+1} is a $A \div$ word if and only if w''_i is a $\div T$ word. Then, since n is even, $w^{2t} = 0$ if and only if $(w'_1 \cdots w'_n)^{2t} = 0$ and $(w''_1 \cdots w''_n)^{2t} = 0$. We may therefore assume in turn that w is a linear combination of $A \div T$ words, or is a linear combination of $T \div A$ words.

We may also assume that each w_i is either a linear combination of $A \div T$ words, or of $T \div A$ words, or of $A \div A$ words, or of $T \div T$ words. We consider these cases in turn. If w_i is a

- $A \div T$ word: then $\psi(w_i) = \begin{pmatrix} u & v \\ u & v \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (u \ v)$ and we set $x_i = u + v$;
- $T \div A$ word: then $\psi(w_i) = \begin{pmatrix} u & u \\ v & v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} (1 \ 1)$ and we set $x_i = u + v$;
- $A \div A$ word: then $\psi(w_i) = \begin{pmatrix} u & u \\ u & u \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(1 \ 1)$ and we set $x_i = u$;
- $T \div T$ word: then $\psi(w_i) = \begin{pmatrix} u & v \\ w & x \end{pmatrix}$ and we set $x_i = u + v + w + x$.

If w is a linear combination of $A \div T$ words, then $\psi(w^{2t})A = (x_1 \cdots x_k)^{2t}A$, and by hypothesis $((x_1 \cdots x_k)^2)^t = 0$, so $w^{2t}A = 0$. If w is a linear combination of $T \div A$ words, then $A\psi(w^{2t}) = 0$ by the same argument. In all cases $w^{2t+1} = 0$. ■

PROPOSITION 4.28: *If $\mathbb{k} = \mathbb{F}_2$, then \mathfrak{A} is graded nil; more precisely, given $x \in \mathfrak{A}$ homogeneous of degree n , we have $x^{72n} = 0$.*

Proof: If n is odd, we may replace x by x^2 , which will be of even degree $2n$. It is therefore sufficient to show that $x^{18n} = 0$ for all homogeneous elements of even degree n , and from now on we assume that n is even.

Assume first that $x \in \mathfrak{A}_1^n$. Then $x = x_1 \cdots x_n$, and since \mathfrak{A}_1 is spanned by $\{A, B, C, D\}$ with $B+C+D = 0$, we may write $x_i = \alpha_i A + \beta_i T_i$ with $\alpha_i, \beta_i \in \mathbb{F}_2$ and $T_i \in \{B, C, D\}$. Furthermore, since n is even, we have $x = x' + x''$ where x' and x'' are monomials, with $x' \in (AT)^{n/2}$ and $x'' \in (TA)^{n/2}$. Therefore $x^8 = (x')^8 + (x'')^8 = 0$ by Proposition 4.21.

In the general case, set $t_0 = 8$ and $t_{i+1} = 2t_i + 1$. Then $t_i = 9 \cdot 2^i - 1$. Find $k \in \mathbb{N}$ such that $n \leq 2^k < 2n$. Then, applying k times Lemma 4.27 to x , we have $x^{t_k} = 0$, so *a fortiori* $x^{18n} = 0$. ■

The following result answers a question in [44]; it also answers a conjecture attributed to Goodearl [15, Conjecture 3.1].

THEOREM 4.29: *If \mathbb{k} is algebraic over \mathbb{F}_2 , then \mathfrak{A} is graded nil and Jacobson radical.*

If \mathbb{k} is not algebraic over \mathbb{F}_2 , then \mathfrak{A} is not graded nil, and it is primitive.

Proof: Assume first that \mathbb{k} is algebraic over \mathbb{F}_2 . Choose a homogeneous $x \in \mathfrak{A}(\mathbb{k})$ of degree ≥ 1 . Then $x \in \mathfrak{A}(\mathbb{F}_{2^n})$ for some n , and therefore x may be seen as a homogeneous element in $M_n(\mathfrak{A})$, by embedding \mathbb{F}_{2^n} as a maximal field in $M_n(\mathbb{F}_2)$. Now \mathfrak{A} is graded nil by Proposition 4.28, so $M_n(\mathfrak{A})$ is graded nil by Lemma 4.25, so $\mathfrak{A}(\mathbb{F}_{2^n})$ is graded nil by restriction, and therefore $\varpi(\mathbb{F}_{2^n}) = \mathfrak{A}(\mathbb{F}_{2^n})$ by Proposition 4.26 and Lemma 3.15; finally $\mathfrak{A}(\mathbb{k})$ is Jacobson radical since it is a union of such algebras.

Assume now that \mathbb{k} contains a transcendental element t . Then \mathfrak{A} is primitive by Proposition 4.23, and the proof of Theorem 4.20, just as Proposition 4.26, implies that \mathfrak{A} is not graded nil. Indeed the element $A + B + Dt$ has infinite order. ■

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